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Non-equilibrium vapour condensation on a shock-tube endwall behind a reflected shock wave

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This paper deals theoretically with a filmwise condensation of a vapour on the endwall of a shock tube behind a reflected shock wave. The gas dynamics, accompanied by heat and mass transfer at the vapour–liquid interface, is treated by the method of matched asymptotic expansions. The first and second approximate solutions are obtained and evaluated numerically. It is clarified that there exists a transition process on the growth of a liquid film, that is, the liquid film grows approximately in proportion to the time t in the early stages after the reflection of the shock wave, and after some time, it grows in proportion to the square root of the time. This transition process from the t-dependent growth to the $t^{\frac{1}{2}}$-dependent one is mainly controlled by the intensity of condensation. In the t-dependent growth region, the growth rate of the liquid film is proportional to the condensation parameter, depending both upon an initial condition and upon thermal properties of the vapour and the liquid film, while in the $t^{\frac{1}{2}}$-dependent growth region it becomes independent of the condensation parameter and is controlled only by thermal properties of the vapour, liquid film and shock-tube endwall. This result suggests that the measurement of the condensation parameter by shock tubes should be made in the t-dependent growth region immediately after the reflection of the shock wave.

1. Introduction

The condensation of a vapour in a thermodynamic non-equilibrium state is one of the fundamental problems in thermo-fluid dynamics and it is also concerned with many problems in engineering. From the aspect of molecular gas dynamics, it has been treated as a problem of the interaction between vapour molecules and their condensed matter. Theoretical studies have been made by Mortensen & Eyring (1960) using the theory of rate processes, and by Schrage (1953), Kogan & Makashev (1971), Sone & Onishi (1973, 1978) and Labuntsov & Kryukov (1979) from the viewpoint of the kinetic theory of gases. Experimental studies, on the other hand, have been made by Wegener & Pouring (1964), Kawada & Mori (1973) and Sislian & Glass (1976) using nozzles or shock tubes.

Of particular relevance to the subject treated in this paper are the shock-tube experiments by Goldstein (1964), Grosse & Smith (1968), Smith (1973) and Fujikawa *et al.* (1982, 1985). Goldstein (1964) first made an experiment on the condensation of water vapour on the sidewalls of a shock tube and tried to measure a so-called condensation coefficient (Schrage 1953; Labuntsov & Kryukov 1979). The condensation was produced in the following way. Behind an incident shock wave, the vapour is compressed and heated, but it is rapidly cooled because of the large difference in heat capacities between the vapour and the shock-tube sidewalls. An unsteady thermal boundary layer forms on the walls and it develops into the vapour region. Under an appropriate initial condition, the vapour begins to condense on the walls. The experiment showed that filmwise or dropwise condensation took place on the sidewalls depending upon their nature. Following Goldstein, Grosse & Smith (1968) also confirmed the condensation of vapour on the sidewalls of a shock tube. The growth of a liquid film formed on the walls was, however, found to be disturbed by the presence of a viscous boundary layer on the walls behind the incident shock wave. Later, Smith (1973) improved Goldstein's method by paying attention to a filmwise condensation on the endwall of a shock tube behind a reflected shock wave. The experiment demonstrated that a liquid film formed uniformly on the endwall and that the growth behaviour of it might be analysed in terms of two different models, one for time less than 10 μs and one for longer times. Except in the early stages after the reflection of the shock wave, the liquid film grew in proportion to the square root of the time for time intervals of the order of milliseconds. Unfortunately, Smith was not able to deduce the condensation coefficient from the experimental data because the growth process of the liquid film mentioned above was not sufficiently understood. Fujikawa *et al.* (1982, 1985) followed the experimental method which Smith proposed (1973), and tried to get the condensation coefficient of methanol vapour, paying attention to an early stage of the liquid-film growth; however, the growth mechanism of the liquid film remained vague. It is therefore important to construct a theoretical framework of the condensation process on the shock-tube endwall and to cast light on it, because Smith's method seems to be quite suitable not only for measuring condensation coefficients of vapours but for understanding other fundamental aspects of condensation – for example, the temperature discontinuity at the liquid surface. It is also expected that the analysis will be of intrinsic interest, because it deals with a general problem in one-dimensional heat and mass transfer among three phases, i.e. vapour, liquid film and solid wall.

It should be mentioned here that the present analysis is the development of a work of Clarke (1967) to the shock-induced condensation, in which he has treated the reflection of a shock wave from a heat-conducting wall by using the method of matched asymptotic expansions.

2. Theoretical analysis

2.1. *Statement of problem*

Figure 1 shows the propagation process of a shock wave in a vapour advancing towards and reflecting then from the solid end wall of a shock tube. O is taken to be the origin of time, the moment of the shock-wave reflection. Just at the instant when the shock wave reflects at the endwall, the pressure, temperature and density of the vapour increase in a stepwise fashion from an initially low state to a high one. The temperature of the vapour at the endwall decreases rapidly because of the large difference in heat capacities between the vapour and the solid endwall. Then, the vapour begins to condense in the form of a liquid film on the endwall and an unsteady thermal boundary layer develops into the vapour region at the same time. The liquid film continues to grow as the time lapses. The vapour flow induced by the condensation has an influence on the propagation process of the reflected shock wave. We will deal theoretically with the gas dynamics accompanied by heat and mass transfer at the vapour–liquid interface.

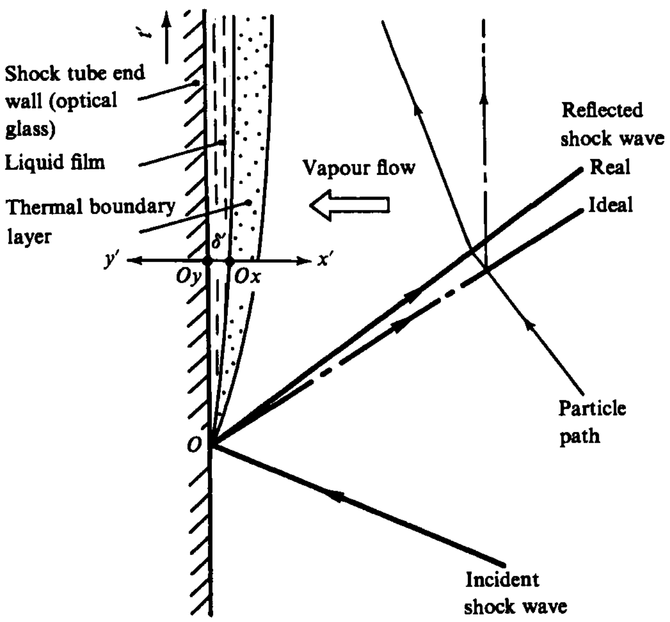

FIGURE 1. The propagation process of the shock wave in the vapour advancing towards and reflecting from the shock-tube endwall.

2.2. *Governing equations*

The following assumptions will be made: (i) the vapour flow is one-dimensional; (ii) the vapour is thermally and calorically perfect; (iii) internal degrees of freedom of vapour molecules are neglected; (iv) a homogeneous nucleation within the thermal boundary layer is neglected; (v) the thermal conductivity and the shear viscosity of the vapour are proportional to the vapour temperature; (vi) the physical properties of liquid film and shock-tube endwall (optical glass) are constant; (vii) a liquid film with a certain thickness already exists on the shock-tube endwall before the incidence of the shock wave. The validity of assumption (i) is restricted by the bifurcation effect due to an interaction of the reflected shock wave with a viscous boundary layer on shock-tube sidewalls. For a shock tube with 30 mm diameter, this assumption may be valid during about 40–50 μs for rather weak shock waves (incident shock Mach number less than 1.2). The assumptions (ii)–(vi) greatly facilitate the analysis without losing the generality of it, but they are not absolutely necessary. The analysis in an ideal situation seems to be the first step in understanding the problem under consideration. Assumption (vii) is reasonable in the case where an initially set vapour pressure, before the reflection of the shock wave, is not too low in comparison with the saturated vapour pressure.

The coordinate system for the vapour, liquid film and shock-tube endwall is also shown in figure 1. O_x is the origin of the x'-coordinate system stemming from the liquid-film surface into the vapour region and it is moving with the growth of the liquid film. However, it is not necessary to describe governing equations for the vapour by the moving coordinate because the growth rate of the liquid film is much smaller than the velocity of the vapour at the liquid-film surface. O_y is the origin of the y'-coordinate system which stems from the liquid–solid interface into the shock-tube endwall. The $\delta'(t')$ indicates the thickness of the liquid film. The position of O_y measured by the x'-system is $-\delta'(t')$. The prime indicates a dimensional quantity.

Governing equations for the vapour described by the x'-coordinate system are given as follows:

continuity equation:
$$\frac{\mathrm{D}\rho'}{\mathrm{D}t'}+\rho'\frac{\partial u'}{\partial x'}=0, \tag{2.1}$$

momentum equation:
$$\rho'\frac{\mathrm{D}u'}{\mathrm{D}t'}+\frac{\partial p'}{\partial x'}=\frac{\partial}{\partial x'}\left\{(\kappa'+\tfrac{4}{3}\mu')\frac{\partial u'}{\partial x'}\right\}, \tag{2.2}$$

energy equation:
$$\rho' C_p'\frac{\mathrm{D}T'}{\mathrm{D}t'}-\frac{Dp'}{Dt'}=\frac{\partial}{\partial x'}\left(\lambda'\frac{\partial T'}{\partial x'}\right)+(\kappa'+\tfrac{4}{3}\mu')\left(\frac{\partial u'}{\partial x'}\right)^2, \tag{2.3}$$

state equation:
$$p'=\rho'\mathscr{R}'T', \tag{2.4}$$

where t' is the time, ρ' the density, p' the pressure, T' the temperature, u' the velocity, C_p' the specific heat of vapour at constant pressure, λ' the thermal conductivity, κ' the bulk viscosity, μ' the shear viscosity and $\mathscr{R}'$ the gas constant. The bulk viscosity κ' will be disregarded. $\mathrm{D}/\mathrm{D}t'$ is the usual convective operator.

The temperatures of the liquid film and the shock-tube endwall satisfy the following heat conduction equations:

$$\frac{\partial T_1'}{\partial t'}-\dot{\delta}'\frac{\partial T_1'}{\partial x'}=D_1'\frac{\partial^2 T_1'}{\partial x'^2}\quad (x'<0), \tag{2.5}$$

$$\frac{\partial T_\mathrm{s}'}{\partial t'}=D_\mathrm{s}'\frac{\partial^2 T_\mathrm{s}'}{\partial y'^2}\quad (y'>0), \tag{2.6}$$

where D_1' and D_s' are the thermal diffusivities of the liquid film and the shock-tube endwall respectively, and they are assumed to be constant. The subscripts l and s indicate the liquid film and the shock-tube endwall. The dot means the differentation with respect to the time t'. The term including $\dot{\delta}'$ in (2.5) indicates the effect of the growth of the liquid film on the heat flow.

The velocity and temperature of the vapour at the liquid-film surface are given by the kinetic theory of gases as follows (Sone & Onishi 1973, 1978):

$$\frac{u'(t',0)}{\{2\mathscr{R}'T_1'(t',0)\}^{\frac{1}{2}}}=-\sigma_\mathrm{c}\left\{\frac{p'(t',0)}{p^{*\prime}(t')}-1\right\}, \tag{2.7}$$

$$\frac{T'(t',0)}{T_1'(t',0)}=1-\sigma_\mathrm{t}\frac{u'(t',0)}{\{2\mathscr{R}'T_1'(t',0)\}^{\frac{1}{2}}}, \tag{2.8}$$

where $p^{*\prime}(t')$ is a saturated vapour pressure corresponding to the surface temperature of the liquid film and the quantities σ_c and σ_t are positive constant values which depend upon the kind of vapour and the surface condition of the liquid film. They can be interpreted as substance parameters and will be called hereinafter the 'condensation parameter' and the 'temperature parameter' respectively. These values must be determined by experiments. In particular, in the case where molecules approaching the liquid surface are completely captured by the surface and the molecules emitted from the surface have a Maxwellian distribution corresponding to the saturated vapour at the temperature of the condensed matter, the condensation parameter σ_c is 4.6904×10^{-1} and the temperature parameter σ_t is 4.4675×10^{-1} (Sone & Onishi 1973, 1978). It must be noted that (2.7) and (2.8) are valid only for cases of low-intensity condensation, i.e. a flow Mach number in the reflected shock region $\ll 1$. The denominator $(2\mathscr{R}'T_1'(t',0))^{\frac{1}{2}}$ of the left-hand side of (2.7) is a quantity with the dimension of the velocity and is of the order of sound speed. Therefore, (2.7) is valid under a condition $|u'(t',0)/(2\mathscr{R}'T_1'(t',0))^{\frac{1}{2}}| \ll 1$. This restriction comes from the fact that (2.7) has been derived from a relaxation model of a linearized Boltzmann equation under the assumption of a small deviation from an equilibrium state. A further remark should be made on (2.7). A so-called condensation coefficient (Schrage 1953; Labuntsov & Kryukov 1979) is not used in (2.7), but instead the condensation parameter is introduced. The value of $\sigma_c = 4.6904 \times 10^{-1}$ just corresponds to the condensation coefficient unity if the vapour density is assumed to be equal to the saturated vapour density in (2.7). The value of σ_c may be actually smaller than the one above.

The energy balance at the liquid-film surface is:

$$\lambda_1'\left(\frac{\partial T_1'}{\partial x'}\right)_{x'=0} = \left(\lambda'\frac{\partial T'}{\partial x'}\right)_{x'=0} - (\rho' u')_{x'=0} L' + \tfrac{4}{3}\left(\mu' u'\frac{\partial u'}{\partial x'}\right)_{x'=0}, \tag{2.9}$$

where L' is the latent heat of condensation. The temperature of the liquid film and the shock-tube endwall and the energy balance at the liquid–solid interface are:

$$T_1'(t', x' = -\delta') = T_s'(t', y' = 0), \tag{2.10}$$

$$\lambda_1'\left(\frac{\partial T_1'}{\partial x'}\right)_{x'=-\delta'} = -\lambda_s'\left(\frac{\partial T_s'}{\partial y'}\right)_{y'=0}. \tag{2.11}$$

The initial conditions are set as follows:

$$p'(0, x' > 0) = p_\infty, \tag{2.12a}$$

$$T'(0, x' > 0) = T_\infty, \tag{2.12b}$$

$$\rho'(0, x' > 0) = \rho_\infty, \tag{2.12c}$$

$$u'(0, x' > 0) = 0, \tag{2.12d}$$

$$T_1'(0, x' < 0) = cT_\infty, \tag{2.13}$$

$$T_s'(0, y' \geqq 0) = cT_\infty. \tag{2.14}$$

The fluid dynamical quantities $(p_\infty, T_\infty, \rho_\infty)$ with the suffix ∞ are the ones in an undisturbed state behind an 'ideal' reflected shock wave. The word 'ideal' means a steady, inviscid, non-heat-conducting and non-vapour-condensing state. Before the initial instant the liquid film and the solid are at a uniform temperature cT_∞.

2.3. *Non-dimensional variables and variable transformations*

The following non-dimensional variables are introduced:

$$\left.\begin{aligned} &p = \frac{p'}{p_\infty}, \quad \theta = \frac{T'}{T_\infty}, \quad \rho = \frac{\rho'}{\rho_\infty}, \quad u = \frac{u'}{a_\infty}, \quad x = \frac{x'}{a_\infty t_\infty}, \quad y = \frac{y'}{(D_s' t_\infty)^{\frac{1}{2}}}, \quad t = \frac{t'}{t_\infty}, \\ &\lambda = \frac{\lambda'}{\lambda_\infty}, \quad \mu = \frac{\mu'}{\mu_\infty}, \quad D = \frac{D'}{D_\infty}, \quad L = \frac{L'}{T_\infty C_p'}, \quad \mathscr{R} = \frac{\mathscr{R}'}{C_p'}, \quad \delta = \frac{\delta'}{a_\infty t_\infty}, \quad m = \frac{m'}{a_\infty \rho_\infty}, \end{aligned}\right\} \tag{2.15}$$

$$Pr = \frac{\mu_\infty C_p'}{\lambda_\infty}, \quad R = \frac{a_\infty^2 t_\infty}{D_\infty}, \tag{2.16}$$

where a_∞, λ_∞, μ_∞ and $D_\infty(=\lambda_\infty/\rho_\infty/C'_p)$ are reference values of the sound speed of the vapour, the thermal conductivity, the shear viscosity and the thermal diffusivity, respectively. Two non-dimensional parameters are introduced, as shown in (2.16), the Prandtl number Pr and a parameter R. The latter can be interpreted as a Reynolds number defined by the sound speed a_∞, the length $a_\infty t_\infty$ and the thermal diffusivity D_∞. It should be noted that there is no macroscopic reference length *a priori* in the problem which we are treating now, so that an arbitrary reference time t_∞ had to be introduced in order to define a reference length. However, it will be shown at a later stage that the reference time or the reference length is uniquely determinable from R through a perturbation matching procedure.

Now, it is convenient to introduce a Lagrangian coordinate χ for the gas dynamic equations:

$$\chi = \int_0^x \rho(t,\tilde{x})\,\mathrm{d}\tilde{x}. \tag{2.17}$$

Hence,

$$\frac{\partial \chi}{\partial x} = \rho, \tag{2.18a}$$

$$\frac{\partial \chi}{\partial t} = \int_0^x \frac{\partial \rho(t,\tilde{x})}{\partial t}\mathrm{d}\tilde{x} = -\int_0^x \frac{\partial}{\partial \tilde{x}}(\rho u)\,\mathrm{d}\tilde{x} = -(\rho u - m), \tag{2.18b}$$

where $m=(\rho u)_{x=0}$, the mass flux rate at the vapour–liquid interface and a function of the time alone. Using χ and t as independent variables, we obtain the following relations (Spalding 1956):

$$\frac{\partial}{\partial x} \to \rho\frac{\partial}{\partial \chi}, \tag{2.19a}$$

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - (\rho u - m)\frac{\partial}{\partial \chi}. \tag{2.19b}$$

For a growing liquid film, a modified Landau transformation will be introduced (Ockendon & Hodgkins 1975):

$$\xi = \frac{x}{\delta} + 1. \tag{2.20}$$

The domain of the space variable is now $[0,1]$, while it was $[-\delta,0]$ in the (x,t)-coordinates. The moving-boundary problem thus becomes a fixed one. Using ξ and t as independent variables, we obtain the following relations:

$$\frac{\partial}{\partial x} \to \frac{1}{\delta}\frac{\partial}{\partial \xi}, \tag{2.21a}$$

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \xi\frac{\dot{\delta}}{\delta}\frac{\partial}{\partial \xi}, \tag{2.21b}$$

where the growth rate of the liquid film $\dot{\delta}$ can be related to the mass flux rate m at the vapour–liquid interface through

$$\dot{\delta} = -\frac{1}{\rho_l}m, \tag{2.22}$$

where the dot means differentiation with respect to the non-dimensional time t and will be used hereinafter in this sense.

By the use of the non-dimensional variables of (2.15), two parameters of (2.16) and the transformations of (2.19*a*, *b*) and (2.21*a*, *b*), (2.1)–(2.14) reduce to

$$u_\chi = \left(\frac{\theta}{p}\right)_t + m\left(\frac{\theta}{p}\right)_\chi, \tag{2.23}$$

$$u_t + mu_\chi + \frac{1}{\gamma}p_\chi = \frac{4}{3}\frac{Pr}{R}(pu_\chi)_\chi, \tag{2.24}$$

$$\theta_t + m\theta_\chi - \left(\frac{\gamma-1}{\gamma}\right)\frac{\theta}{p}(p_t + mp_\chi) = \frac{1}{R}(p\theta_\chi)_\chi + \frac{4}{3}\frac{Pr}{R}(\gamma-1)\,p(u_\chi)^2, \tag{2.25}$$

$$p = \rho\theta \quad (p = \rho\lambda = \rho\mu), \tag{2.26}$$

$$\theta_{1t} - \xi\frac{\dot{\delta}}{\delta}\theta_{1\xi} = \frac{1}{R}\frac{D_1}{\delta^2}\theta_{1\xi\xi}, \tag{2.27}$$

$$\theta_{st} = \theta_{syy}, \tag{2.28}$$

$$u(t,0) = -\sigma_c\left\{\frac{2\theta_1(t,1)}{\gamma}\right\}^{\frac{1}{2}}\left\{\frac{p(t,0)}{p^*(t)} - 1\right\}, \tag{2.29}$$

$$\theta(t,0) = \theta_1(t,1)\left[1 + \sigma_c\,\sigma_t\left\{\frac{p(t,0)}{p^*(t)} - 1\right\}\right], \tag{2.30}$$

$$(\theta_{1\xi})_{\xi=1} = \frac{\delta}{\lambda_1}\{\rho_1 LR\dot{\delta} + p(t,0)\,\theta_\chi(t,0) + \tfrac{4}{3}\gamma Pr\mathscr{R}\theta(t,0)\,u(t,0)\,u_\chi(t,0)\}, \tag{2.31}$$

$$\theta_1(t,\xi=0) = \theta_s(t,y=0), \tag{2.32}$$

$$(\theta_{sy})_{y=0} = -\frac{\lambda_1}{\lambda_s}\left(\frac{D_s}{R}\right)^{\frac{1}{2}}\frac{1}{\delta}(\theta_{1\xi})_{\xi=0}, \tag{2.33}$$

$$p(0,\chi) = 1, \quad \theta(0,\chi) = 1, \quad \rho(0,\chi) = 1, \quad u(0,\chi) = 0, \tag{2.34}$$

$$\theta_1(0,\xi < 1) = c, \tag{2.35}$$

$$\theta_s(0,y \geqslant 0) = c, \tag{2.36}$$

where γ is the specific heat ratio of the vapour and $p_t, p_\chi, \ldots$ etc. denote partial differentiations with respect to $t, \chi, \ldots$ (except subscripts l and s).

2.4. *Matched asymptotic expansions*

Clarke (1967) has made the analysis of shock-wave reflection from a heat-conducting shock-tube endwall by using the method of matched asymptotic expansions. We will follow the work of Clarke, in which a thin thermal boundary layer is introduced in the gas region near the interface and the whole gas region is divided into two parts, an 'inner' one and an 'outer' one.

2.4.1. *Outer expansions*

The pressure, temperature and velocity of the vapour behind the reflected shock wave are assumed to be expanded as follows.

$$p(t,\chi) \sim p_0(t,\chi) + \epsilon_1(R)\,p_1(t,\chi) + \ldots, \tag{2.37}$$

$$\theta(t,\chi) \sim \theta_0(t,\chi) + \epsilon_1(R)\,\theta_1(t,\chi) + \ldots, \tag{2.38}$$

$$u(t,\chi) \sim u_0(t,\chi) + \epsilon_1(R)\,u_1(t,\chi) + \ldots, \tag{2.39}$$

which are valid as $R \to \infty$ with t, χ fixed. The functions p_n, θ_n and u_n $(n = 0, 1, 2, \ldots)$ are of the order of unity. The functions $\epsilon_n(R)$ must be determined so as to ensure this, and $\epsilon_{n+1}(R)/\epsilon_n(R) \to 0$ as $R \to \infty$. We assume that $\epsilon_0(R) = 1$. It is reasonable to assume that the leading terms in (2.37)–(2.39) are put equal to the quantities for the 'ideal' reflected shock wave given by (2.34), i.e.

$$p_0(t,\chi) = 1, \quad \theta_0(t,\chi) = 1, \quad \rho_0(t,\chi) = 1, \quad u_0(t,\chi) = 0. \tag{2.40}$$

2.4.2. *Inner expansions*

The pressure, temperature and velocity of the vapour adjacent to the liquid-film surface are assumed to be expanded by stretching the χ-coordinate, i.e.

$$p(t,\chi) \sim P_0(t,\psi) + \Delta_1(R)\, P_1(t,\psi) + \ldots, \tag{2.41}$$

$$\theta(t,\chi) \sim \Theta_0(t,\psi) + \Delta_1(R)\, \Theta_1(t,\psi) + \ldots, \tag{2.42}$$

$$u(t,\chi) \sim \Delta_0(R)\{U_0(t,\psi) + \Delta_1(R)\, U_1(t,\psi) + \ldots\}. \tag{2.43}$$

The functions P_n, Θ_n $(n = 0, 1, 2, \ldots)$ are of the order of unity; the functions $\Delta_n(R)$ must be determined so as to ensure this. Concerning the velocity, the normalizing function $\Delta_0(R)$ is introduced as shown in (2.43) and it can be determined by the matching between the boundary condition (2.29) and (2.43) at the vapour–liquid interface. We might reasonably suppose that the liquid film and shock-tube endwall temperatures (θ_1, θ_s) and the liquid-film thickness δ are closely connected to the inner solutions for the vapour under consideration through (2.29), (2.30) and (2.32). Therefore, these functions are assumed to be expanded in the forms

$$\theta_1(t,\xi) \sim \theta_{10}(t,\xi) + \Delta_1(R)\, \theta_{11}(t,\xi) + \ldots, \tag{2.44}$$

$$\theta_s(t,y) \sim \theta_{s0}(t,y) + \Delta_1(R)\, \theta_{s1}(t,y) + \ldots, \tag{2.45}$$

$$\delta(t) \sim \Delta_0(R)\{\delta_0(t) + \Delta_1(R)\, \delta_1(t) + \ldots\}. \tag{2.46}$$

The functions θ_{1n}, θ_{sn} and δ_n $(n = 0, 1, 2, \ldots)$ are of the order of unity. It is convenient to expand the mass flux rate m formally as follows.

$$m(t) \sim \Delta_0(R)\{m_0(t) + \Delta_1(R)\, m_1(t) + \ldots\} \tag{2.47}$$

where the function m_n $(n = 0, 1, 2, \ldots)$ is of the order of unity. This expression is reasonable from the relations with (2.22) and (2.46).

Substituting (2.41)–(2.47) into (2.23)–(2.25), (2.27), (2.31) and (2.33), we find that it is quite evident that the only reasonable choices for the stretched variable ψ and the normalizing function $\Delta_0(R)$ are

$$\psi = R^{\frac{1}{2}}\chi, \quad \Delta_0(R) = 1/R^{\frac{1}{2}}. \tag{2.48}$$

The parameter R in (2.48) is arbitrary at this stage because there is an arbitrary reference time t_∞. Let us show here that the parameter R is uniquely determinable from the kind of vapour and an initial condition. From (2.29) and (2.43), we obtain

$$\begin{aligned} u(t,0) &= \Delta_0(R)\{U_0(t,0) + \Delta_1(R)\, U_1(t,0) + \ldots\} \\ &= \Delta_0(R)\, U_0(t,0) + \Delta_1(R)\, \Delta_0(R)\, U_1(t,0) + \ldots \\ &= -\sigma_c \left\{\frac{2\theta_{10}(t,1)}{\gamma}\right\}^{\frac{1}{2}} \left\{\frac{P_0(t,0)}{p_0^*(t)} - 1\right\} - \Delta_1(R)\, \sigma_c \left\{\frac{2\theta_{10}(t,1)}{\gamma}\right\}^{\frac{1}{2}} \frac{1}{p_0^*(t)} \\ &\quad \times \left\{P_1(t,0) - \frac{P_0(t,0)}{p_0^*(t)}\, p_1^*(t) + \frac{P_0(t,0) - p_0^*(t)}{2\theta_{10}(t,1)}\, \theta_{11}(t,1)\right\} + \ldots, \end{aligned}$$

where $p^*(t)$ has been expanded in the form of (2.41). Therefore, we obtain

$$\varDelta_0(R)\,U_0(t,0) = -\sigma_{\mathrm{c}}\left\{\frac{2\theta_{10}(t,1)}{\gamma}\right\}^{\frac{1}{2}}\left\{\frac{P_0(t,0)}{p_0^*(t)}-1\right\}, \tag{2.49a}$$

$$\varDelta_0(R)\,U_1(t,0) = -\sigma_{\mathrm{c}}\left\{\frac{2\theta_{10}(t,1)}{\gamma}\right\}^{\frac{1}{2}}\frac{1}{p_0^*(t)}\left\{P_1(t,0)-\frac{P_0(t,0)}{p_0^*(t)}p_1^*(t) + \frac{P_0(t,0)-p_0^*(t)}{2\theta_{10}(t,1)}\theta_{11}(t,1)\right\}. \tag{2.49b}$$

Noting that U_0 ($U_0 < 0$ for condensation) and $(2\theta_{10}/\gamma)^{\frac{1}{2}}$ are of the order of unity, we can reasonably choose $\varDelta_0$ or R as follows.

$$\varDelta_0 = \sigma_{\mathrm{c}}\left\{\frac{P_0(0,0)}{p_0^*(0)}-1\right\}, \tag{2.50}$$

or

$$R = \left[\frac{p_0^*(0)}{\sigma_{\mathrm{c}}\{P_0(0,0)-p_0^*(0)\}}\right]^2. \tag{2.51}$$

It should be noted that the parameters $\varDelta_0$ and R depend upon the condensation parameter (σ_{c}) and an initial condition ($P_0(0,0), p_0^*(0)$). The functional form of R has now been explicitly given; however, R is still arbitrary. The restriction imposed upon R should be determined from the enlargement factor $R^{\frac{1}{2}}$ of the variable χ. It is reasonable to impose the restriction $R^{\frac{1}{2}} \gg 1$ upon the parameter R. Therefore, the initial condition $(P_0(0,0)-p_0^*(0))/p_0^*(0)$ must be chosen so as to be much smaller than unity in the case where σ_{c} is near the maximum value ($= 4.6904\times10^{-1}$); on the other hand, it can be chosen arbitrarily so as to satisfy the restriction $R^{\frac{1}{2}} \gg 1$ in the case where σ_{c} is very small. The first inner solution of the flow velocity at the vapour–liquid interface is given by

$$U_0(t,0) = -\left\{\frac{2\theta_{10}(t,1)}{\gamma}\right\}^{\frac{1}{2}}\frac{p_0^*(0)}{p_0^*(t)}\,\frac{P_0(t,0)-p_0^*(t)}{P_0(0,0)-p_0^*(0)}. \tag{2.52}$$

Similarly, the second inner solution $U_1(t,0)$ can be obtained from (2.49*b*).

The governing equations concerning the first inner approximation (subscript 0) are:

$$U_{0\psi} = \left(\frac{\Theta_0}{P_0}\right)_t + m_0\left(\frac{\Theta_0}{P_0}\right)_\psi, \tag{2.53}$$

$$P_{0\psi} = 0, \tag{2.54}$$

$$\Theta_{0t} + m_0\,\Theta_{0\psi} - \left(\frac{\gamma-1}{\gamma}\right)\frac{\Theta_0}{P_0}(P_{0t}+m_0\,P_{0\psi}) = (P_0\,\Theta_{0\psi})_\psi, \tag{2.55}$$

$$\theta_{10t} - \xi\frac{\dot{\delta}_0}{\delta_0}\theta_{10\xi} = \frac{D_1}{\delta_0^2}\theta_{10\xi\xi}, \tag{2.56}$$

$$\theta_{s0t} = \theta_{s0yy}, \tag{2.57}$$

where the mass flux rate m_0 is given as

$$m_0 = \left(\frac{P_0\,U_0}{\Theta_0}\right)_{\psi=0} \tag{2.58a}$$

$$= -\left\{\frac{2\theta_{10}(t,1)}{\gamma}\right\}^{\frac{1}{2}}\frac{p_0^*(0)}{p_0^*(t)}\,\frac{P_0(t,0)}{\Theta_0(t,0)}\,\frac{P_0(t,0)-p_0^*(t)}{P_0(0,0)-p_0^*(0)}. \tag{2.58b}$$

Matching one-term inner and one-term outer expansions by using the asymptotic matching principle (Van Dyke 1964), we have

$$p_0(t,0) = P_0(t,\infty), \quad \theta_0(t,0) = \Theta_0(t,\infty). \tag{2.59}$$

Equations (2.40), (2.54) and (2.59) yield

$$P_0(t,\psi) = 1, \tag{2.60}$$

$$\Theta_0(t,\infty) = 1. \tag{2.61}$$

Equation (2.60) indicates that the vapour pressure in the first approximation is spatially uniform and that it is the constant value behind the 'ideal' reflected shock wave. Equations (2.53), (2.55) and (2.60) yield

$$U_{0\psi} = \Theta_{0t} + m_0 \Theta_{0\psi}, \tag{2.62}$$

$$\Theta_{0t} + m_0 \Theta_{0\psi} = \Theta_{0\psi\psi}. \tag{2.63}$$

Similarly, substituting (2.41)–(2.47) into (2.30)–(2.33), (2.22), (2.35) and (2.36) we obtain the boundary and initial conditions

$$\Theta_0(t,0) = \theta_{l0}(t,1)\left[1 + \sigma_c \sigma_t \left\{\frac{1}{p_0^*(t)} - 1\right\}\right], \tag{2.64}$$

$$(\theta_{l0\xi})_{\xi=1} = \frac{\delta_0}{\lambda_l}\{\rho_l L \dot{\delta}_0 + \Theta_{0\psi}(t,0)\}, \tag{2.65}$$

$$\theta_{l0}(t,\xi=0) = \theta_{s0}(t,y=0), \tag{2.66}$$

$$(\theta_{s0y})_{y=0} = -\frac{\lambda_l}{\lambda_s}\frac{(D_s)^{\frac{1}{2}}}{\delta_0}(\theta_{l0\xi})_{\xi=0}, \tag{2.67}$$

$$\dot{\delta}_0 = -\frac{1}{\rho_l} m_0, \tag{2.68}$$

$$\theta_{l0}(0,\xi<1) = c, \tag{2.69}$$

$$\theta_{s0}(0,y \geqslant 0) = c. \tag{2.70}$$

2.5. *Temperatures in vapour, liquid film, shock-tube endwall and velocity in vapour: first approximation*

We will first attempt to analyse the temperature in the vapour. The temperature field of the vapour is described by (2.63). There is no initial condition for Θ_0 and it cannot be found from the matching technique. Based on the discussion by Clarke (1967), we set

$$\Theta_0(0,\psi) = 1. \tag{2.71}$$

It is consistent with the boundary condition (2.61) at $\psi = \infty$. The boundary condition at $\psi = 0$ is given by (2.64). In the case $m_0 = 0$, (2.63) becomes a simple heat-conduction equation and it can be easily solved (Carslaw & Jaeger 1959). In the case $m_0 \neq 0$, however, it is not possible to analytically solve this equation by standard methods. Beylich (1985) has treated this type of equation, concerning the thermodynamics of a single bubble, employing the Laplace transformation in the sense of distributions. We will follow here the method which Beylich used. If we define ϕ as the Laplace transformation of Θ_0 with respect to ψ,

$$\phi(t,p) = \mathscr{L}_{\psi}\{\Theta_0(t,\psi)\} = \int_0^{\infty} \mathrm{e}^{-p\psi}\Theta_0(t,\psi)\,\mathrm{d}\psi, \tag{2.72}$$

we obtain an ordinary differential equation which the function ϕ satisfies,

$$\phi_t + p(m_0 - p)\phi = T_{w0}(m_0 - p) - T'_{w0}, \tag{2.73}$$

where p is the complex variable (not the pressure here), $T_{w0} = \Theta_0(t, 0)$, and $T'_{w0} = \Theta_{0\psi}(t, 0)$ the gradient of the vapour temperature with respect to ψ at the liquid surface. The solution of (2.73) is

$$\phi(t,p) = \phi^0 \exp\left[p^2 t - p\int_0^t m_0(t')\,\mathrm{d}t'\right] + \int_0^t [T_{w0}(t')\{m_0(t') - p\} - T'_{w0}(t')] \exp\left[p^2(t-t') - p\int_{t'}^t m_0(\xi)\,\mathrm{d}\xi\right]\mathrm{d}t', \tag{2.74}$$

where

$$\phi^0 = \mathscr{L}_\psi\{1\}.$$

Performing the inverse Laplace transformation to (2.74), defined as

$$\Theta_0(t,\psi) = \mathscr{L}_\psi^{-1}\{\phi(t,p)\} = \frac{1}{2\pi \mathrm{i}}\int_{q-\mathrm{i}\infty}^{q+\mathrm{i}\infty} \phi(t,p)\,\mathrm{e}^{\psi p}\,\mathrm{d}p, \quad q = \mathrm{Re}\,(p), \tag{2.75}$$

we finally obtain a solution of (2.63) as

$$\begin{aligned}\Theta_0(t,\psi) = {} & \tfrac{1}{2}\mathrm{erfc}\left[-\frac{1}{2t^{\frac{1}{2}}}\left\{\psi - \int_0^t m_0(\zeta)\,\mathrm{d}\zeta\right\}\right] \\ & + \frac{1}{2\pi^{\frac{1}{2}}}\int_0^t \frac{T_{w0}(t')\,m_0(t') - T'_{w0}(t')}{(t-t')^{\frac{1}{2}}} \exp\left[-\frac{1}{4(t-t')}\left\{\psi - \int_{t'}^t m_0(\zeta)\,\mathrm{d}\zeta\right\}^2\right]\mathrm{d}t' \\ & + \frac{1}{2\pi^{\frac{1}{2}}}\int_0^t \frac{T_{w0}(t')}{2(t-t')^{\frac{3}{2}}}\left\{\psi - \int_{t'}^t m_0(\zeta)\,\mathrm{d}\zeta\right\} \exp\left[-\frac{1}{4(t-t')}\left\{\psi - \int_{t'}^t m_0(\zeta)\,\mathrm{d}\zeta\right\}^2\right]\mathrm{d}t'.\end{aligned} \tag{2.76}$$

Equation (2.76) satisfies the boundary condition (2.61), i.e.

$$\lim_{\psi\to\infty} \Theta_0(t,\psi) = 1,$$

as it must do. The vapour temperature at the liquid surface can then be obtained by taking the limit $\psi \to 0$ in (2.76), i.e.

$$\begin{aligned}T_{w0}(t) = {} & \mathrm{erfc}\left[\frac{1}{2t^{\frac{1}{2}}}\int_0^t m_0(\zeta)\,\mathrm{d}\zeta\right] \\ & - \frac{1}{\pi^{\frac{1}{2}}}\int_0^t \frac{T'_{w0}(t')}{(t-t')^{\frac{1}{2}}} \exp\left[-\frac{1}{4(t-t')}\left\{\int_{t'}^t m_0(\zeta)\,\mathrm{d}\zeta\right\}^2\right]\mathrm{d}t' \\ & + \frac{1}{\pi^{\frac{1}{2}}}\int_0^t \left[m_0(t') - \frac{1}{2(t-t')}\int_{t'}^t m_0(\zeta)\,\mathrm{d}\zeta\right]\frac{T_{w0}(t')}{(t-t')^{\frac{1}{2}}} \exp\left[-\frac{1}{4(t-t')}\left\{\int_{t'}^t m_0(\zeta)\,\mathrm{d}\zeta\right\}^2\right]\mathrm{d}t'.\end{aligned} \tag{2.77}$$

Equation (2.77) is an integral equation for the temperature gradient T'_{w0} with the boundary condition (2.64). From (2.62), (2.63) and (2.68), we obtain the solution of the velocity of the vapour near the liquid surface,

$$U_0(t,\psi) = \Theta_{0\psi}(t,\psi) - T'_{w0}(t) - \rho_1\,\dot{\delta}_0\,T_{w0}(t) \tag{2.78}$$

where

$$\begin{aligned}\Theta_{0\psi}(t,\psi) = {} & \frac{1}{2(\pi t)^{\frac{1}{2}}}\exp\left[-\frac{1}{4t}\left\{\psi-\int_0^t m_0(\zeta)\,\mathrm{d}\zeta\right\}^2\right] \\ & -\frac{1}{2\pi^{\frac{1}{2}}}\int_0^t \frac{T_{\mathrm{w0}}(t')\,m_0(t')-T'_{\mathrm{w0}}(t')}{2(t-t')^{\frac{3}{2}}}\left\{\psi-\int_{t'}^t m_0(\zeta)\,\mathrm{d}\zeta\right\} \\ & \times\exp\left[-\frac{1}{4(t-t')}\left\{\psi-\int_{t'}^t m_0(\zeta)\,\mathrm{d}\zeta\right\}^2\right]\mathrm{d}t' \\ & +\frac{1}{2\pi^{\frac{1}{2}}}\int_0^t T_{\mathrm{w0}}(t')\left[\frac{1}{2(t-t')^{\frac{3}{2}}}-\frac{1}{4(t-t')^{\frac{5}{2}}}\left\{\psi-\int_{t'}^t m_0(\zeta)\,\mathrm{d}\zeta\right\}^2\right] \\ & \times\exp\left[-\frac{1}{4(t-t')}\left\{\psi-\int_{t'}^t m_0(\zeta)\,\mathrm{d}\zeta\right\}^2\right]\mathrm{d}t', \end{aligned} \tag{2.79}$$

$$\begin{aligned}T'_{\mathrm{w0}}(t) = {} & \frac{1}{(\pi t)^{\frac{1}{2}}}\exp\left[-\frac{1}{4t}\left\{\int_0^t m_0(\zeta)\,\mathrm{d}\zeta\right\}^2\right] \\ & -\frac{1}{\pi^{\frac{1}{2}}}\int_0^t \frac{1}{(t-t')^{\frac{1}{2}}}\frac{\partial}{\partial t'}\left\{T_{\mathrm{w0}}(t')\exp\left[-\frac{1}{4(t-t')}\left\{\int_{t'}^t m_0(\zeta)\,\mathrm{d}\zeta\right\}^2\right]\right\}\mathrm{d}t' \\ & +\frac{1}{2\pi^{\frac{1}{2}}}\int_0^t \frac{T_{\mathrm{w0}}(t')\,m_0(t')-T'_{\mathrm{w0}}(t')}{(t-t')^{\frac{3}{2}}}\int_{t'}^t m_0(\zeta)\,\mathrm{d}\zeta\exp\left[-\frac{1}{4(t-t')}\left\{\int_{t'}^t m_0(\zeta)\,\mathrm{d}\zeta\right\}^2\right]\mathrm{d}t' \\ & -\frac{1}{2\pi^{\frac{1}{2}}}\int_0^t \frac{T_{\mathrm{w0}}(t')}{2(t-t')^{\frac{5}{2}}}\left\{\int_{t'}^t m_0(\zeta)\,\mathrm{d}\zeta\right\}^2\exp\left[-\frac{1}{4(t-t')}\left\{\int_{t'}^t m_0(\zeta)\,\mathrm{d}\zeta\right\}^2\right]\mathrm{d}t'. \end{aligned} \tag{2.80}$$

From (2.78), we obtain

$$U_0(t,0) = -\rho_1\,\dot{\delta}_0\,T_{\mathrm{w0}}(t), \tag{2.81}$$

$$U_0(t,\infty) = -T'_{\mathrm{w0}}(t)-\rho_1\,\dot{\delta}_0\,T_{\mathrm{w0}}(t). \tag{2.82}$$

Equations (2.81) and (2.82) show that the flow velocity of the vapour at the liquid surface is smaller than that in the outer region because the high-temperature vapour in the thermal boundary layer is cooled and the density consequently becomes large.

Next, let us proceed to the analysis of the temperature in the liquid film. The governing equation of the liquid film is rewritten below:

$$\theta_{10t}-\xi\frac{\dot{\delta}_0}{\delta_0}\theta_{10\xi} = \frac{D_1}{\delta_0^2}\theta_{10\xi\xi}, \tag{2.83}$$

for which the boundary and initial conditions are given by (2.65), (2.66) and (2.69). We will introduce here a similarity variable

$$z = \xi F(t), \tag{2.84}$$

where F is an unknown function which should be sought. Using this similarity variable as an independent variable, (2.83) is transformed to

$$\frac{z\theta'_{10}}{\theta''_{10}} = \frac{D_1 F^3}{\delta_0^2(\dot{F}-\dot{\delta}_0\,\delta_0^{-1}F)} \tag{2.85a}$$

$$= A, \tag{2.85b}$$

where the prime denotes differentiation with respect to z. The left-hand side of (2.85*a*) is a function of the variable z alone; on the other hand, the right-hand side is a function of the variable t alone. Therefore, (2.85*a*) must be equal to a constant value, which will be written as A, as in (2.85*b*). We then obtain from (2.85*a*, *b*):

$$\dot{F}-\frac{\dot{\delta}_0}{\delta_0}F-\frac{D_1}{A\delta_0^2}F^3=0, \tag{2.86a}$$

$$\theta''_{10}-\frac{z}{A}\theta'_{10}=0. \tag{2.86b}$$

The solution of (2.86*a*) is

$$F(t)=\pm\frac{F(0)\,\delta_0(t)}{\{-2D_1\,F^2(0)\,t/A+\delta_0^2(0)\}^{\frac{1}{2}}}, \tag{2.87}$$

where $\delta_0(0)$ and $F(0)$ are initial values of δ_0 and F, respectively, and they must be determined by experiment. The constant A must be negative so that the denominator of (2.87) is real at all times. Noting that $A < 0$, θ'_{10} in (2.86*b*) is expected to be a function of an exponential type,

$$\theta'_{10}=\exp{(-z^2)}. \tag{2.88}$$

Substituting (2.88) into (2.86*b*), we obtain $A=-\frac{1}{2}$. Finally, we obtain a general solution of (2.86*b*) as

$$\theta_{10}=c_1\int^z\exp{(-u^2)}\,\mathrm{d}u+c_2, \tag{2.89}$$

where c_1 and c_2 are constants of integration, which can be determined from the boundary conditions (2.65) and (2.66). Equation (2.89) then reduces to

$$\theta_{10}(t,\xi)=\theta_{s0}(t,y=0)+G(t)\,\mathrm{erf}\,[\xi F(t)], \tag{2.90}$$

where

$$G(t)=\frac{\pi^{\frac{1}{2}}\exp{(F^2)}}{2F}(\theta_{10\xi})_{\xi=1}.$$

The temperature of the liquid film at the vapour–liquid interface is given at $\xi=1$ in (2.90) to be

$$\theta_{10}(t,\xi=1)=\theta_{s0}(t,y=0)+G(t)\,\mathrm{erf}\,[F(t)]. \tag{2.91}$$

Equations (2.90) and (2.91) hold for F both with positive and negative signs, because $\mathrm{erf}\,(-z)=-\mathrm{erf}\,(z)$ and because of the above definition of the function G. The function F with a positive sign in (2.87) will be adopted hereafter. As $A=-\frac{1}{2}$,

$$F(t)=\frac{F(0)\,\delta_0(t)}{\{4D_1\,F^2(0)\,t+\delta_0^2(0)\}^{\frac{1}{2}}}. \tag{2.92}$$

The initial value $F(0)\,(\geqslant 0)$ of the function F in (2.92) can be determined in the following way. Taking the limit of (2.91) as $t\rightarrow 0$, we obtain

$$\lim_{t\rightarrow 0}\theta_{10}(t,\xi=1)=\lim_{t\rightarrow 0}\theta_{s0}(t,y=0)+\lim_{t\rightarrow 0}G(t)\,\mathrm{erf}\,[F(t)] \tag{2.93a}$$

$$=c+\Delta\theta_{10}(t=0,\xi=1), \tag{2.93b}$$

where $\Delta\theta_{10}(t=0,\xi=1)$ is an impulsive temperature variation of the liquid film surface just at the instant of reflection of the shock wave. However, there is no information about $\Delta\theta_{10}$ in the present theoretical framework and the only way for determining it is by experiment. A discussion will be given here by treating $\Delta\theta_{10}$ as

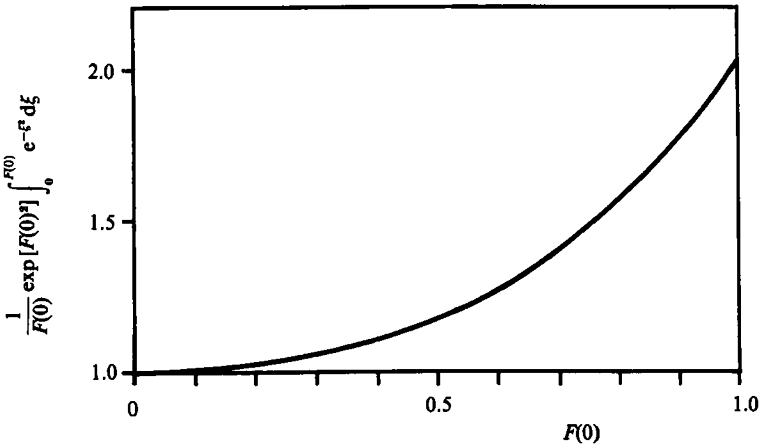

FIGURE 2. The graph of (2.94) versus $F(0)$.

a known quantity. It is physically reasonable to assume that it is positive for condensation. Using the boundary condition (2.65) and (2.93a,b), we obtain

$$\frac{1}{F(0)}\exp[F^2(0)]\int_0^{F(0)} \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi = \frac{\lambda_1\,\Delta\theta_{10}(t=0,\xi=1)}{\delta_0(0)\{\rho_1 L\dot{\delta}_0(0)+\Theta_{0\psi}(0,0)\}}. \tag{2.94}$$

The initial value of the function F can be determined numerically from (2.94), because the right-hand side of this equation is now known. The left-hand side increases monotonically from unity as $F(0)$ changes from 0 to a finite value. The graph of (2.94) is shown in figure 2. The final value of F as $t\rightarrow\infty$ is, on the other hand,

$$\lim_{t\rightarrow\infty} F(t) = \lim_{t\rightarrow\infty}\frac{F(0)\,\delta_0(t)}{\{4D_1\,F^2(0)\,t+\delta_0^2(0)\}^{\frac{1}{2}}} \tag{2.95a}$$

$$\doteqdot \frac{\lambda_s\{\theta_{10}(t\rightarrow\infty,\xi=1)-c\}}{\rho_1\,L(\pi D_1\,D_s)^{\frac{1}{2}}}, \tag{2.95b}$$

where $\theta_{10}(t\rightarrow\infty,\xi=1)=$ a constant value, the final temperature of the liquid film surface. Equation (2.95b) will be proved in §4. For a condition of an incident-shock Mach number less than 1.2, $F(t\rightarrow\infty)\approx O(10^{-2})$. It is found, therefore, that the influence of F on the temperature of the liquid film surface is very small.

The temperature in the endwall of the shock tube can be obtained by solving (2.57) with the boundary and initial conditions (2.67) and (2.70), and it is given by

$$\theta_{s0}(t,y) = c-\frac{1}{\pi^{\frac{1}{2}}}\int_0^t \frac{(\theta_{s0y})_{y=0}}{(t-t')^{\frac{1}{2}}}\exp\left[-\frac{y^2}{4(t-t')}\right]\mathrm{d}t', \tag{2.96}$$

where $$(\theta_{s0y})_{y=0} = -\frac{\lambda_1}{\lambda_s}\frac{(D_s)^{\frac{1}{2}}}{\delta_0}\exp(F^2)\,(\theta_{10\xi})_{\xi=1}.$$

The temperature of the endwall surface can be estimated at $y=0$ as

$$\theta_{s0}(t,0) = c-\frac{1}{\pi^{\frac{1}{2}}}\int_0^t \frac{(\theta_{s0y})_{y=0}}{(t-t')^{\frac{1}{2}}}\mathrm{d}t'. \tag{2.97}$$

2.6. *Unsteady process of reflection of shock wave: second approximation*

We will now proceed to the analysis of the reflection process of the shock wave from the heat-conducting and phase-changing surface. The second approximation for the outer expansion is treated here. Matching a two-term outer series and a one-term inner series for the vapour velocity shows that $\epsilon_1(R)$ should be put equal to $R^{-\frac{1}{2}}$. Then, we have

$$u_1(t,0) = U_0(t,\infty). \tag{2.98}$$

Substituting (2.37)–(2.39) and (2.47) into (2.23)–(2.25), we obtain equations for the second-order terms, with suffix 1,

$$u_{1\chi} = \theta_{1t} - p_{1t} + V_0(\theta_{1\chi} - p_{1\chi}), \tag{2.99}$$

$$\gamma(u_{1t} + V_0 u_{1\chi}) + p_{1\chi} = 0, \tag{2.100}$$

$$\gamma(\theta_{1t} + V_0 \theta_{1\chi}) = (\gamma-1)(p_{1t} + V_0 p_{1\chi}), \tag{2.101}$$

where

$$V_0 = -\sigma_c \left\{\frac{2}{\gamma}\theta_{10}(t,1)\right\}^{\frac{1}{2}} \frac{1}{\Theta_0(t,0)} \left\{\frac{1}{p_0^*(t)} - 1\right\}.$$

Rearranging these equations, we obtain

$$(1-V_0^2)\,p_{1\chi\chi} = p_{1tt} + V_0\{2p_{1\chi t} + p_{1\chi}(\ln V_0)^{\cdot}\}, \tag{2.102}$$

$$(1-V_0^2)\,u_{1\chi\chi} = u_{1tt} + V_0\{2u_{1\chi t} + u_{1\chi}(\ln V_0)^{\cdot}\}. \tag{2.103}$$

Expanding the functions p_1 and u_1 as

$$p_1(t,\chi) \sim p_{10}(t,\chi) + p_{11}(t,\chi) + \dots, \tag{2.104}$$

$$u_1(t,\chi) \sim u_{10}(t,\chi) + u_{11}(t,\chi) + \dots, \tag{2.105}$$

and noting that $|V_0| \ll 1$, $p_{11} \sim O(V_0)$ and $u_{11} \sim O(V_0)$, from (2.102) and (2.103), we have

$$p_{10\chi\chi} = p_{10tt}, \tag{2.106}$$

$$u_{10\chi\chi} = u_{10tt}, \tag{2.107}$$

$$p_{11\chi\chi} = p_{11tt} + V_0\{2p_{10\chi t} + p_{10\chi}(\ln V_0)^{\cdot}\}, \tag{2.108}$$

$$u_{11\chi\chi} = u_{11tt} + V_0\{2u_{10\chi t} + u_{10\chi}(\ln V_0)^{\cdot}\}. \tag{2.109}$$

It may be sufficient to get the solutions only for the first-order terms (p_{10} and u_{10}) because $|V_0| \ll 1$. They can be obtained from the wave equations (2.106) and (2.107). The boundary conditions can be found at the front of the reflected shock wave. Let U_s represent the speed of the reflected shock wave relative to the vapour ahead of it. The location of χ_s (the distance from the vapour–liquid interface ($\chi = 0$)) can be written as

$$\chi_s = \frac{\rho_a a_a}{\rho_\infty a_\infty} \int_0^t M_s(\tilde{t})\,\mathrm{d}\tilde{t}, \tag{2.110}$$

where

$$M_s(t) = \frac{U_s(t)}{a_a}, \tag{2.111}$$

and the subscript a denotes a quantity behind the incident shock wave. The pressure, density and velocity of the vapour behind the reflected shock wave are given by Rankine–Hugoniot relations as follows:

$$p(t,\chi_s)=\frac{p_a}{p_\infty}\left[1+\frac{2\gamma}{\gamma+1}\{M_s^2(t)-1\}\right], \tag{2.112}$$

$$\rho(t,\chi_s)=\frac{\rho_a}{\rho_\infty}\left[1-\frac{2}{\gamma+1}\left\{1-\frac{1}{M_s^2(t)}\right\}\right]^{-1}, \tag{2.113}$$

$$u(t,\chi_s)=-\frac{u_a}{a_\infty}+\frac{2}{\gamma+1}\frac{a_a}{a_\infty}\left\{M_s(t)-\frac{1}{M_s(t)}\right\}. \tag{2.114}$$

M_s and χ_s are assumed to be expanded in a series in the perturbation parameter $\epsilon_n(R)$, i.e.

$$M_s(t)=M_{s0}+\epsilon_1(R)\,M_{s1}(t)+\dots, \tag{2.115}$$

$$\chi_s(t)=\chi_{s0}+\epsilon_1(R)\frac{\rho_a a_a}{\rho_\infty a_\infty}\int_0^t M_{s1}(\tilde{t})\,\mathrm{d}\tilde{t}+\dots, \tag{2.116}$$

where χ_{s0} represents the location of a so-called 'ideal' reflected shock wave, i.e.

$$\chi_{s0}=\left(\frac{\rho_a a_a}{\rho_\infty a_\infty}\right)M_{s0}\,t \tag{2.117a}$$

$$=\alpha t\quad\left(\alpha=\left(\frac{\rho_a a_a}{\rho_\infty a_\infty}\right)M_{s0}\right). \tag{2.117b}$$

Equations (2.37)–(2.39), (2.112)–(2.115) and the state equation (2.26) yield

$$p_1(t,\chi_{s0})=\left(\frac{4\gamma}{\gamma+1}\right)\left(\frac{p_a}{p_\infty}\right)M_{s0}M_{s1}(t), \tag{2.118}$$

$$\theta_1(t,\chi_{s0})=p_1(t,\chi_{s0})-\left(\frac{4}{\gamma+1}\right)\left(\frac{\rho_\infty}{\rho_a}\right)\frac{M_{s1}(t)}{M_{s0}^3}, \tag{2.119}$$

$$u_1(t,\chi_{s0})=\left(\frac{2}{\gamma+1}\right)\left(\frac{a_a}{a_\infty}\right)\left(1+\frac{1}{M_{s0}^2}\right)M_{s1}(t). \tag{2.120}$$

The solutions of the wave equations (2.106) and (2.107) with the boundary conditions (2.118) and (2.120) are

$$p_{10}(t,\chi)=\gamma H(t-\chi)-\gamma\left(\frac{1-\Phi}{1+\Phi}\right)H\left(\left[\frac{1-\alpha}{1+\alpha}\right][t+\chi]\right), \tag{2.121}$$

$$u_{10}(t,\chi)=H(t-\chi)+\left(\frac{1-\Phi}{1+\Phi}\right)H\left(\left[\frac{1-\alpha}{1+\alpha}\right][t+\chi]\right), \tag{2.122}$$

where
$$\Phi=\frac{2\alpha}{1+M_{s0}^{-2}}.$$

Making use of (2.101), (2.119) and noting that $|V_0|\ll 1$, we obtain

$$\theta_1(t,\chi)=\left(\frac{\gamma-1}{\gamma}\right)p_1(t,\chi)+\frac{2\Phi}{1+\Phi}\left(1-\frac{1}{\alpha^2M_{s0}^2}\right)H\left(\left[\frac{1-\alpha}{\alpha}\right]\chi\right). \tag{2.123}$$

The unknown function H in (2.121)–(2.123) can be determined from the matching condition (2.98) for the vapour velocity. Letting (2.122) match (2.82) at $\chi = 0$, we obtain

$$H(t)+\left(\frac{1-\Phi}{1+\Phi}\right)H\left(\left[\frac{1-\alpha}{1+\alpha}\right]t\right) = -T'_{\mathrm{w0}}(t)-\rho_1\dot{\delta}_0 T_{\mathrm{w0}}(t). \tag{2.124}$$

Because $(1-\Phi)/(1+\Phi) \ll 1$ for all permitted reflected shock Mach numbers (Clarke 1967), the function H can be reasonably approximated by

$$H(t) \doteqdot -T'_{\mathrm{w0}}(t)-\rho_1\dot{\delta}_0 T_{\mathrm{w0}}(t). \tag{2.125}$$

The correction function M_{s1} of the reflected shock Mach number is then obtained from (2.120), (2.122) and (2.125) to be

$$M_{\mathrm{s1}}(t) \doteqdot -\tfrac{1}{2}(\gamma+1)\left(\frac{a_\infty}{a_a}\right)\left(\frac{M^2_{\mathrm{s0}}}{1+M^2_{\mathrm{s0}}}\right)\{T'_{\mathrm{w0}}([1-\alpha]t)+\rho_1\dot{\delta}_0([1-\alpha]t)\,T_{\mathrm{w0}}([1-\alpha]t)\}. \tag{2.126}$$

The first term in the braces of the right-hand side of (2.126) represents the effect of heat conduction on the reflected shock Mach number, and the second term the effect of condensation on it.

2.7. *Temperatures in vapour, liquid film, shock-tube endwall and velocity in vapour: second approximation*

Matching two-term inner and two-term outer expansions shows that $\Delta_1(R)$ should be put equal to $R^{-\frac{1}{2}}$. Therefore, matching conditions are

$$P_1(t,\infty) = p_1(t,0), \quad \Theta_1(t,\infty) = \theta_1(t,0). \tag{2.127\,a,b}$$

A set of equations satisfied by the second inner terms is obtained by substituting (2.41)–(2.47) into (2.23)–(2.25), (2.27), (2.28), (2.30)–(2.33), (2.22), (2.35) and (2.36):

$$U_{1\psi} = \Theta_{1t}+m_0\,\Theta_{1\psi}-\Theta_{0t}\,P_1-\Theta_0\,P_{1t}-m_0\,\Theta_{0\psi}P_1+m_1\,\Theta_{0\psi}, \tag{2.128}$$

$$P_{1\psi} = 0, \tag{2.129}$$

$$\Theta_{1t}+m_0\,\Theta_{1\psi}-\Theta_{1\psi\psi} = P_1\,\Theta_{0\psi\psi}+\left(\frac{\gamma-1}{\gamma}\right)\Theta_0\,P_{1t}-m_1\,\Theta_{0\psi}, \tag{2.130}$$

$$\theta_{11t}-\xi\frac{\dot{\delta}_0}{\delta_0}\theta_{11\xi} = \frac{D_1}{\delta_0^2}\theta_{11\xi\xi}+\xi\left(\frac{\delta_1}{\delta_0}\right)^{\boldsymbol{\cdot}}\theta_{10\xi}-2D_1\frac{\delta_1}{\delta_0^3}\theta_{10\xi\xi}, \tag{2.131}$$

$$\theta_{\mathrm{s1}t} = \theta_{\mathrm{s1}yy}, \tag{2.132}$$

$$\Theta_1(t,0) = \theta_{11}(t,1)+\sigma_{\mathrm{c}}\,\sigma_{\mathrm{t}}\left[\theta_{11}(t,1)\left\{\frac{1}{p_0^*(t)}-1\right\}+\frac{\theta_{10}(t,1)}{p_0^*(t)}\left\{P_1(t,0)-\frac{p_1^*(t)}{p_0^*(t)}\right\}\right], \tag{2.133}$$

$$(\theta_{11\xi})_{\xi=1} = \frac{1}{\lambda_1}[\rho_1\,L(\delta_0\,\delta_1)^{\boldsymbol{\cdot}}+\{\delta_0\,P_1(t,0)+\delta_1\}\,T'_{\mathrm{w0}}(t)+\delta_0\,T'_{\mathrm{w1}}(t)], \tag{2.134}$$

$$\theta_{11}(t,\xi=0) = \theta_{\mathrm{s1}}(t,y=0), \tag{2.135}$$

$$(\theta_{\mathrm{s1}y})_{y=0} = -\frac{\lambda_1}{\lambda_{\mathrm{s}}}\frac{(D_{\mathrm{s}})^{\frac{1}{2}}}{\delta_0}\left\{(\theta_{11\xi})_{\xi=0}-\frac{\delta_1}{\delta_0}(\theta_{10\xi})_{\xi=0}\right\}, \tag{2.136}$$

$$\dot{\delta}_1 = -\frac{1}{\rho_1}m_1, \tag{2.137}$$

$$\theta_{11}(0,\xi<1) = 0, \tag{2.138}$$

$$\theta_{\mathrm{s1}}(0,y\geqslant 1) = 0, \tag{2.139}$$

where

$$m_1 = -\left\{\frac{2}{\gamma}\theta_{10}(t,1)\right\}^{\frac{1}{2}} \frac{1}{\varTheta_0(t,0)} \frac{p_0^*(0)}{1-p_0^*(0)} \left[\left\{\frac{1}{p_0^*(t)}-1\right\}\left\{\frac{1}{2}\frac{\theta_{11}(t,1)}{\theta_{10}(t,1)}-\frac{\varTheta_1(t,0)}{\varTheta_0(t,0)}\right\} + P_1(t,0)\left\{\frac{1}{p_0^*(t)}-1\right\}+\frac{1}{p_0^*(t)}\left\{P_1(t,0)-\frac{p_1^*(t)}{p_0^*(t)}\right\}\right].$$

Equation (2.129) shows that the correction function P_1 for the vapour pressure depends on the time alone. From (2.121) and (2.127*a*), P_1 is given as

$$P_1(t,\psi) = \gamma H(t) - \gamma\left(\frac{1-\varPhi}{1+\varPhi}\right) H\left(\left[\frac{1-\alpha}{1+\alpha}\right]t\right) \tag{2.140a}$$

$$\doteqdot \gamma H(t). \tag{2.140b}$$

The correction function for the vapour temperature can be obtained from (2.130), with the initial and boundary conditions

$$\varTheta_1(0,\psi) = \theta_1(0,0), \tag{2.141}$$

$$T_{w1}(t) = \varTheta_1(t,0). \tag{2.142}$$

The initial condition for $\varTheta_1$ cannot be exactly determined for the same reason as in the first approximation, but it is quite natural to adopt (2.141) by making use of (2.127*b*). It is extremely difficult to obtain an explicit solution of (2.130) because it includes unknown functions P_1 and m_1 in the inhomogeneous term. However, (2.130) can be solved formally by treating the inhomogeneous term as a known function. Thus, we obtain

$$\begin{aligned}\varTheta_1(t,\psi) = {}& \tfrac{1}{2}\theta_1(0,0)\operatorname{erfc}\left[-\frac{1}{2t^{\frac{1}{2}}}\left\{\psi-\int_0^t m_0(\zeta)\,\mathrm{d}\zeta\right\}\right] \\ & + \frac{1}{2\pi^{\frac{1}{2}}}\int_0^t \frac{T_{w1}(t')\,m_0(t')-T'_{w1}(t')}{(t-t')^{\frac{1}{2}}} \exp\left[-\frac{1}{4(t-t')}\left\{\psi-\int_{t'}^t m_0(\zeta)\,\mathrm{d}\zeta\right\}^2\right]\mathrm{d}t' \\ & + \frac{1}{2\pi^{\frac{1}{2}}}\int_0^t \frac{T_{w1}(t')}{2(t-t')^{\frac{3}{2}}}\left\{\psi-\int_{t'}^t m_0(\zeta)\,\mathrm{d}\zeta\right\} \exp\left[-\frac{1}{4(t-t')}\left\{\psi-\int_{t'}^t m_0(\zeta)\,\mathrm{d}\zeta\right\}^2\right]\mathrm{d}t' \\ & + \frac{1}{2\pi^{\frac{1}{2}}}\int_0^t I(t',\psi) * \frac{1}{(t-t')^{\frac{1}{2}}} \exp\left[-\frac{1}{4(t-t')}\left\{\psi-\int_{t'}^t m_0(\zeta)\,\mathrm{d}\zeta\right\}^2\right]\mathrm{d}t',\end{aligned} \tag{2.143}$$

where $T'_{w1} = \varTheta_{1\psi}(t,0)$, a correction function for the gradient of the vapour temperature at the liquid film surface, and

$$I(t,\psi) = P_1\,\varTheta_{0\psi\psi} + \left(\frac{\gamma-1}{\gamma}\right)\varTheta_0\,P_{1t} - m_1\,\varTheta_{0\psi}.$$

Taking the limit $\psi \to 0$ in (2.143), we obtain an integral equation for T'_{w1},

$$\begin{aligned}T_{w1}(t) \doteqdot {}& \theta_1(0,0)\operatorname{erfc}\left[\frac{1}{2t^{\frac{1}{2}}}\left\{\int_0^t m_0(\zeta)\,\mathrm{d}\zeta\right\}\right] \\ & + \frac{\gamma-1}{\gamma}\int_0^t T_{w0}(t')\,P_{1t}(t')\operatorname{erfc}\left[\frac{1}{2(t-t')^{\frac{1}{2}}}\left\{\int_{t'}^t m_0(\zeta)\,\mathrm{d}\zeta\right\}\right]\mathrm{d}t' \\ & - \frac{1}{\pi^{\frac{1}{2}}}\int_0^t \frac{T'_{w1}(t')+T'_{w0}(t')\,P_1(t')}{(t-t')^{\frac{1}{2}}} \exp\left[-\frac{1}{4(t-t')}\left\{\int_{t'}^t m_0(\zeta)\,\mathrm{d}\zeta\right\}^2\right]\mathrm{d}t' \\ & + \frac{1}{\pi^{\frac{1}{2}}}\int_0^t \frac{T_{w1}(t')}{(t-t')^{\frac{1}{2}}}\left[m_0(t')-\frac{1}{2(t-t')}\int_{t'}^t m_0(\zeta)\,\mathrm{d}\zeta\right] \exp\left[-\frac{1}{4(t-t')}\left\{\int_{t'}^t m_0(\zeta)\,\mathrm{d}\zeta\right\}^2\right]\mathrm{d}t'.\end{aligned} \tag{2.144}$$

The correction function for the liquid temperature can be obtained from (2.131) with the boundary conditions (2.134) and (2.135). Using the similarity variable z defined by (2.84), we obtain

$$\theta''_{11}+2z\theta'_{11} = J(z), \tag{2.145}$$

where

$$J(z) = -\frac{\delta_0^2}{D_1 F^2}\left\{z\left(\frac{\delta_1}{\delta_0}\right)^{\cdot}\theta'_{10}-2D_1\frac{\delta_1}{\delta_0^3}F^2\theta''_{10}\right\},$$

and the prime denotes differentiation with respect to z; the variables ξ and z are defined in the domains $[0, 1]$ and $[0, F]$, respectively. A general solution of (2.145) is

$$\theta_{11}(z) = \theta_{11\mathrm{p}}(z)\left[c_3+\int^z\frac{1}{\theta_{11\mathrm{p}}^2(u_1)\exp(u_1^2)}\left\{c_4+\int^{u_1}J(\tilde{u})\,\theta_{11\mathrm{p}}(\tilde{u})\exp(\tilde{u}^2)\,\mathrm{d}\tilde{u}\right\}\mathrm{d}u_1\right], \tag{2.146}$$

where c_3 and c_4 are constants of integration, and $\theta_{11\mathrm{p}}$ is a particular solution to the homogeneous part of (2.145), given by

$$\theta_{11\mathrm{p}}(z) = \int^z \exp(-u^2)\,\mathrm{d}u. \tag{2.147}$$

The constants c_3 and c_4 are determined from the boundary conditions (2.134) and (2.135) as

$$c_3 = \exp(F^2)\left[\frac{(\theta_{11\xi})_{\xi=1}}{F}-\theta_{11\mathrm{p}}(F)\,K(F)\right]-\int^F K(u)\,\mathrm{d}u,$$

$$c_4 = -\theta_{\mathrm{s}1}(t, y=0), \tag{2.148}$$

where

$$K(u) = \frac{1}{\theta_{11\mathrm{p}}^2(u)\exp(u^2)}\int^u J(\tilde{u})\,\theta_{11\mathrm{p}}(\tilde{u})\exp(\tilde{u}^2)\,\mathrm{d}\tilde{u}.$$

At the vapour–liquid interface ($\xi = 1$), the function θ_{11} can be estimated by putting $z = F$ in (2.146).

Finally, the correction function $\theta_{\mathrm{s}1}$ for the shock-tube endwall temperature is obtained from (2.132) with the boundary and initial conditions (2.136) and (2.139), and is

$$\theta_{\mathrm{s}1}(t,y) = -\frac{1}{\pi^{\frac{1}{2}}}\int_0^t\frac{(\theta_{\mathrm{s}1y})_{y=0}}{(t-t')^{\frac{1}{2}}}\exp\left[-\frac{y^2}{4(t-t')}\right]\mathrm{d}t', \tag{2.149}$$

where

$$(\theta_{\mathrm{s}1y})_{y=0} = -\frac{\lambda_1}{\lambda_{\mathrm{s}}}\frac{(D_{\mathrm{s}})^{\frac{1}{2}}}{\delta_0}\left\{c_3 F-\frac{\delta_1}{\delta_0}\exp(F^2)\,(\theta_{10\xi})_{\xi=1}\right\}.$$

At the liquid–solid interface, the correction function $\theta_{\mathrm{s}1}$ can be estimated by putting $y = 0$ in (2.149).

3. Remarks on numerical computation

Remarks on numerical computation of integral equations (2.77) and (2.144) are given in this section. These equations are used for getting $T'_{\mathrm{w}0}$ and $T'_{\mathrm{w}1}$ which are necessary for the computation of (2.76), (2.79) and (2.143).

The integral equations (2.77) and (2.144) have singularities in the denominators of the memory integrals. The computation of the memory integrals in these equations is made in the following way. For simplicity, let us consider here a typical example,

$$Y = \int_0^t\frac{X(t,t')}{(t-t')^{\frac{1}{2}}}\mathrm{d}t'. \tag{3.1}$$

The right-hand side of (3.1) may be discretized as

$$Y \doteqdot \sum_{k=1}^{i-1} \int_{t_k}^{t_{k+1}} \frac{X(t_i, t')}{(t_i - t')^{\frac{1}{2}}} \mathrm{d}t', \tag{3.2}$$

where $t_1 = 0$ and $t_i = t$. It should be noted that the above integral varies as the time $t(= t_i)$ elapses because the integrand is a function of the time. The function X will be approximated by a straight line within a small time interval $[t_n, t_{n+1}]$, i.e.

$$X(t, t') \doteqdot X_k + \Delta X_{k+1} \cdot (t' - t_k), \tag{3.3}$$

where $X_k = X(t, t_k)$, $\Delta X_k = \Delta X(t, t_k)$ and

$$\Delta X_{k+1} = \frac{X_{k+1} - X_k}{t_{k+1} - t_k}.$$

Using (3.3), (3.2) is then reduced to

$$Y \doteqdot Z(X) + \tfrac{4}{3} X_i \cdot (t - t_{i-1})^{\frac{1}{2}}, \tag{3.4}$$

where

$$\begin{aligned} Z(X) = 2 \sum_{k=1}^{i-2} [&(X_k - \Delta X_{k+1} \cdot t_k)\{(t - t_k)^{\frac{1}{2}} - (t - t_{k+1})^{\frac{1}{2}}\} \\ &+ \tfrac{1}{3} \Delta X_{k+1} \{(2t + t_k)(t - t_k)^{\frac{1}{2}} - (2t + t_{k+1})(t - t_{k+1})^{\frac{1}{2}}\}] + \tfrac{2}{3} X_{i-1} \cdot (t - t_{i-1})^{\frac{1}{2}}. \end{aligned}$$

The memory integral Y has been divided into two parts; one of them can be evaluated by known quantities and the other still contains an unknown quantity X_i.

We will apply the above numerical method for the memory integral (3.1) to (2.77). Defining $X^{(1)}$ and $X^{(2)}$ as

$$X^{(1)}(t, t') = T'_{\mathrm{w0}}(t') \exp[-\omega^2(t, t')], \tag{3.5}$$

$$X^{(2)}(t, t') = T_{\mathrm{w0}}(t')\{m_0(t') - \tfrac{1}{2}\overline{m}_0(t, t')\} \exp[-\omega^2(t, t')], \tag{3.6}$$

(2.77) can be expressed as

$$T_{\mathrm{w0}}(t) = \operatorname{erfc}[\omega(t, 0)] - \frac{1}{\pi^{\frac{1}{2}}} \int_0^t \frac{X^{(1)}(t, t')}{(t - t')^{\frac{1}{2}}} \mathrm{d}t' + \frac{1}{\pi^{\frac{1}{2}}} \int_0^t \frac{X^{(2)}(t, t')}{(t - t')^{\frac{1}{2}}} \mathrm{d}t', \tag{3.7}$$

where

$$\omega(t, t') = \frac{1}{2(t - t')^{\frac{1}{2}}} \int_{t'}^{t} m_0(\zeta)\, \mathrm{d}\zeta,$$

$$\overline{m}_0(t, t') = \frac{1}{t - t'} \int_{t'}^{t} m_0(\zeta)\, \mathrm{d}\zeta.$$

Equations (3.5) and (3.6) diverge at $t' = t$; therefore, (3.7) also diverges at $t' = t$. However, noting that

$$\lim_{t' \to t} \omega(t, t') = 0,$$

$$\lim_{t' \to t} \overline{m}_0(t, t') = m_0(t),$$

we obtain

$$X_i^{(1)} = \lim_{t' \to t} X^{(1)} = T'_{\mathrm{w0}}(t), \tag{3.8}$$

$$X_i^{(2)} = \lim_{t' \to t} X^{(2)} = \tfrac{1}{2} m_0(t)\, T_{\mathrm{w0}}(t). \tag{3.9}$$

The singularity at $t' = t$ of (3.5) and (3.6) has now been removed. Using (3.4), (3.7)–(3.9) and putting $\Delta t \equiv t_i - t_{i-1}$ = constant, we finally obtain the temperature gradient T'_{w0} of the vapour at the liquid surface as

$$T'_{w0}(t) = \left[\operatorname{erfc}\{\omega(t,0)\} - \frac{1}{\pi^{\frac{1}{2}}}\{Z(X^{(1)}) - Z(X^{(2)})\} + \left\{\frac{2}{3}\left(\frac{\Delta t}{\pi}\right)^{\frac{1}{2}} m_0(t) - 1\right\} T_{w0}(t) \right] \Big/ \frac{4}{3}\left(\frac{\Delta t}{\pi}\right)^{\frac{1}{2}}, \tag{3.10}$$

where the vapour temperature T_{w0} at the liquid interface in (3.10) is given by (2.64). The timestep Δt must be determined by trial and error. Tentative numerical results showed that the above procedure was sufficiently accurate for $\Delta t < 0.1$. Equation (2.144) can be solved by the same method as above.

Finally, we should remark on the computation of (2.76) and (2.143) for temperature distributions. For simplicity, let us consider the special case $m_0 = 0$. Then, (2.76) reduces to

$$\Theta_0(t,\psi) = \frac{1}{2}\left[1 - \frac{1}{\pi^{\frac{1}{2}}} \int_0^t \frac{T'_{w0}(t')}{(t-t')^{\frac{1}{2}}} \exp\left\{-\frac{\psi^2}{4(t-t')}\right\} \mathrm{d}t'\right] + \frac{1}{2}\left[\operatorname{erf}\left\{\frac{\psi}{2t^{\frac{1}{2}}}\right\} + \frac{1}{2\pi^{\frac{1}{2}}} \int_0^t \frac{T_{w0}(t')\,\psi}{(t-t')^{\frac{3}{2}}} \exp\left\{-\frac{\psi^2}{4(t-t')}\right\} \mathrm{d}t'\right]. \tag{3.11}$$

In the limit $\psi \rightarrow \infty$, (3.11) approaches unity, as it must do. In the limit $\psi \rightarrow 0$, on the other hand, the right-hand side of (3.11) apparently approaches $\frac{1}{2}T_{w0}$, although it should approach T_{w0} if (3.11) is the correct solution of (2.63) for $m_0 = 0$. This comes from the peculiar (δ-function-like) character of the memory integral in the second term of the right-hand side of (3.11),

$$\frac{1}{2\pi^{\frac{1}{2}}} \int_0^t \frac{T_{w0}(t')\,\psi}{(t-t')^{\frac{3}{2}}} \exp\left\{-\frac{\psi^2}{4(t-t')}\right\} \mathrm{d}t'.$$

Defining a function as

$$g(t-t',\psi) = \frac{1}{2\pi^{\frac{1}{2}}} \frac{\psi}{(t-t')^{\frac{3}{2}}} \exp\left\{-\frac{\psi^2}{4(t-t')}\right\}, \tag{3.12}$$

we find that the function $g(t-t',\psi)$ has the following character in the limit $\psi \rightarrow 0$:

(i) $g(t-t',\psi) \rightarrow 0 \quad (t \neq t')$,
(ii) $g(t-t',\psi) \rightarrow \infty \quad (t = t')$,
(iii) $\int_0^t g(t-t',\psi)\,\mathrm{d}t' \rightarrow 1$.

These properties are similar to those of a Dirac δ-function. Figure 3 shows the function $g(t-t',\psi)$. It is therefore found that (3.11) approaches T_{w0} in the limit $\psi \rightarrow 0$, as expected. Due to the δ-function-like character of the function g, numerical difficulties take place in the neighbourhood of $\psi = 0$ because of the finite value of the space step in the computation. The temperature deviates from the actual one near the interface. The temperature distribution is here estimated near the interface by an interpolation between the surface temperature (given) and a temperature a little distance from the interface. The numerical procedure for (2.143) is the same as the one above.

4. Numerical results and discussion

Numerical computations have been performed for a methanol vapour by running the condensation parameter σ_c through the values: 4.6904×10^{-1}, 4.6904×10^{-2}, 4.6904×10^{-3}. The temperature parameter σ_t is assumed to be 4.4675×10^{-1} (Sone & Onishi 1973, 1978). The initial conditions prior to the incidence of the shock wave

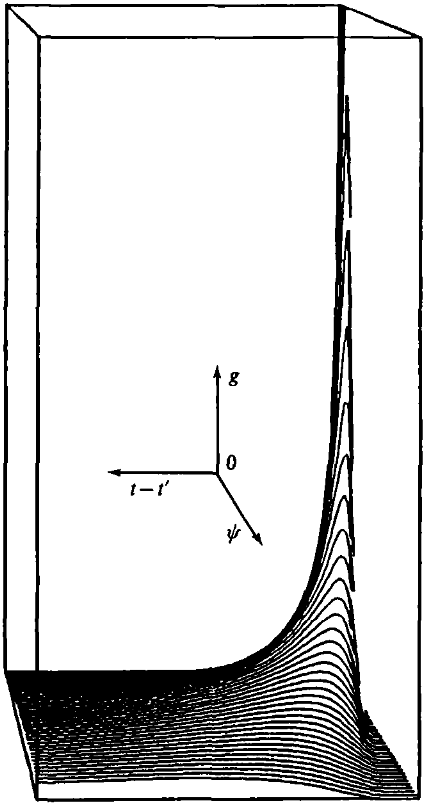

FIGURE 3. The graph of function $g(t-t', \psi)$ defined by (3.12).

on the endwall of a shock tube are: vapour, liquid film and endwall temperatures 293.15 K; vapour pressure 9.7285×10^3 Pa (80 % of the saturated vapour pressure at 293.15 K); incident-shock Mach number 1.15. These conditions are chosen so that the boundary conditions for the vapour velocity and the temperature at the liquid film surface can remain valid throughout the condensation process in the reflected shock region. The physical properties of vapour, liquid film and shock-tube endwall (optical glass: crown) are shown in table 1. The initial thickness of the liquid film is found to have no large influence on the gas dynamics and the consequent growth of the liquid film; therefore, it has been assumed here to be 0.01 μm.

Figure 4 shows the time history of the surface temperatures of the liquid film and the shock-tube endwall. The origin of the time is taken at the moment of the shock-wave reflection. The temperature is normalized by $T_\infty = 331.18$ K, the temperature in the 'ideal' reflected shock region. The broken line indicates the first approximation of the perturbation solution and the solid line the second approximation. The initial value $F(0)$ of the function F (equation (2.92)) is assumed to be 0.01: it will be shown at the end of this section that the final value $F(t \to \infty)$ is about 0.025. Thus, the effect of F on the surface temperatures of the liquid film and the shock-tube endwall is found to be very small. The difference between the first and second approximations is also very small; for example, it is less than 1.5 °C even for $\sigma_c = 4.6904 \times 10^{-1}$ (figure 4(*a*, *b*)). The first approximate solutions are therefore adequate for the estimation of the liquid film and the endwall surface temperatures.

Vapour (methanol)		
Specific heat at constant pressure (C'_p)	1.28×10^{3}	J/(kgK)
Latent heat of condensation (L')	1.10×10^{6}	J/kg
Gas constant ($\mathscr{R}'$)	2.56×10^{2}	J/(kgK)
Thermal conductivity (λ')	$\sim1.68\times10^{-2}$	W/(mK),
Specific heat ratio (γ)	1.25	
Liquid (methanol)		
Thermal diffusivity (D'_l)	1.06×10^{-7}	m²/s
Thermal conductivity (λ'_l)	2.11×10^{-1}	W/(mK)
Density (ρ'_l)	7.93×10^{2}	kg/m³
Solid (crown glass)		
Thermal diffusivity (D'_s)	$\sim5.74\times10^{-7}$	m²/s
Thermal conductivity (λ'_s)	~1.05	W/(mK)

TABLE 1. Physical properties of the vapour, liquid film and shock-tube endwall (optical glass: crown) at approximately 20 °C.

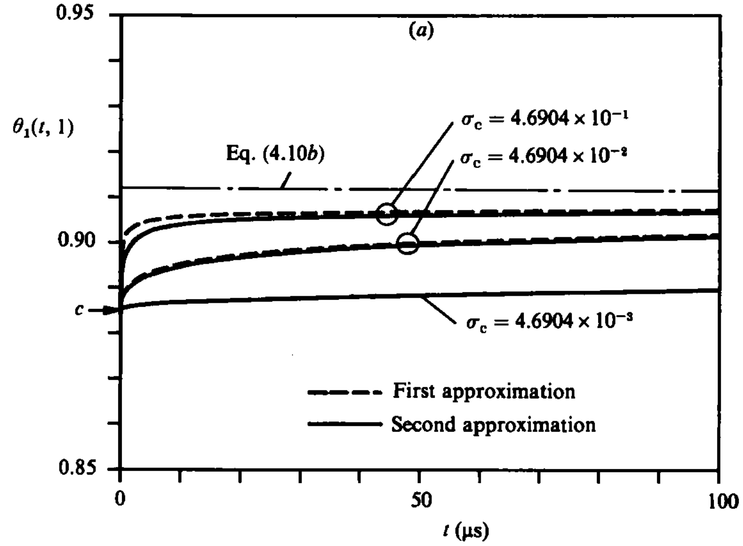

FIGURE 4(*a*). For caption see next page.

The larger the value of σ_c, the more rapid and the larger become the temperature rises of the liquid film and the endwall surfaces because of the larger release of latent heat of condensation. The surface temperature of the liquid film is slightly higher than that of the endwall. The difference between them is about 0.4 °C even at the time of 100 μs for $\sigma_c = 4.6904\times10^{-1}$, and also negligibly small for 4.6904×10^{-3}. This indicates that the thermal condition of the liquid film is governed by that of the shock-tube endwall.

Figure 5 shows the time history of the vapour temperature at the liquid film surface. The difference between the approximations is small; therefore, the first approximation is adequate for the estimation of the vapour temperature. In the case $\sigma_c = 4.6904\times10^{-1}$, the temperature changes greatly within a few microseconds and attains a nearly constant value ($\doteqdot$ 302 K), while in the case of the smaller values,

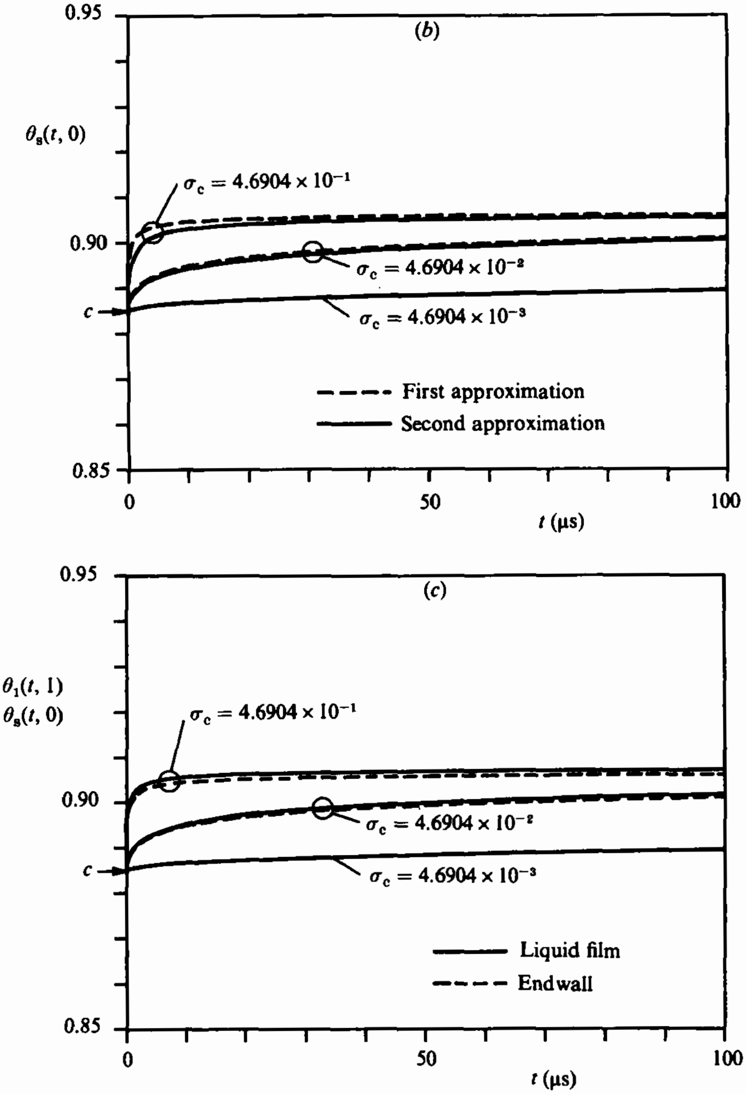

FIGURE 4. The time history of the surface temperatures of the liquid film and the shock-tube endwall ((*a*) liquid-film surface temperature, (*b*) endwall surface temperature and (*c*) comparison between liquid film and endwall surface temperatures (second approximation)).

it drops rapidly and then continues to increase gradually as the surface temperature of the liquid film rises. The abrupt and discontinuous temperature drop occurs for all σ_c at the instant of the reflection of the shock wave, but its structure cannot be treated in the framework of the present perturbation analysis because it takes place on the timescale of the collision time of vapour molecules. The condensation under consideration is a phenomenon which occurs on a much longer timescale. The larger the value of σ_c, the higher the vapour temperature, because it is proportional to the flow velocity of vapour at the liquid surface.

Figure 6 shows the temperature distributions in the vapour, liquid film and shock-tube endwall: (*a*) $\sigma_c = 4.6904 \times 10^{-1}$, (*b*) $\sigma_c = 4.6904 \times 10^{-2}$ and (*c*) $\sigma_c = 4.6904 \times 10^{-3}$. The computation is stopped 10 μs after the reflection of the shock

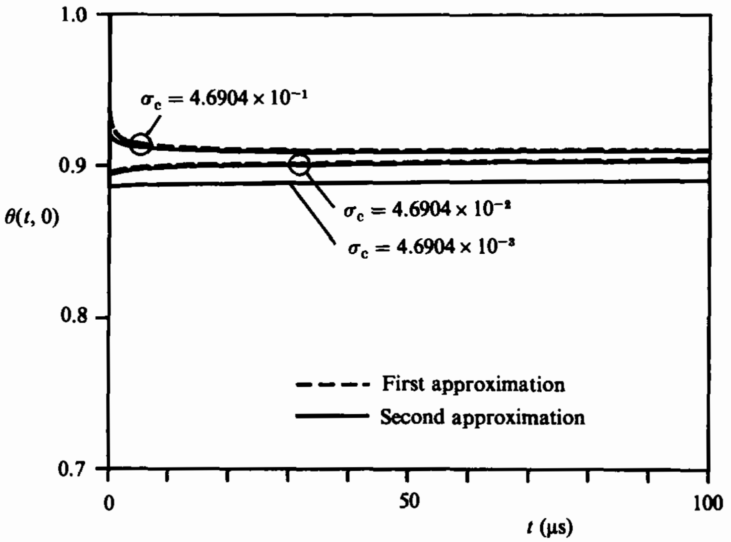

FIGURE 5. The time history of the vapour temperature at the liquid-film surface.

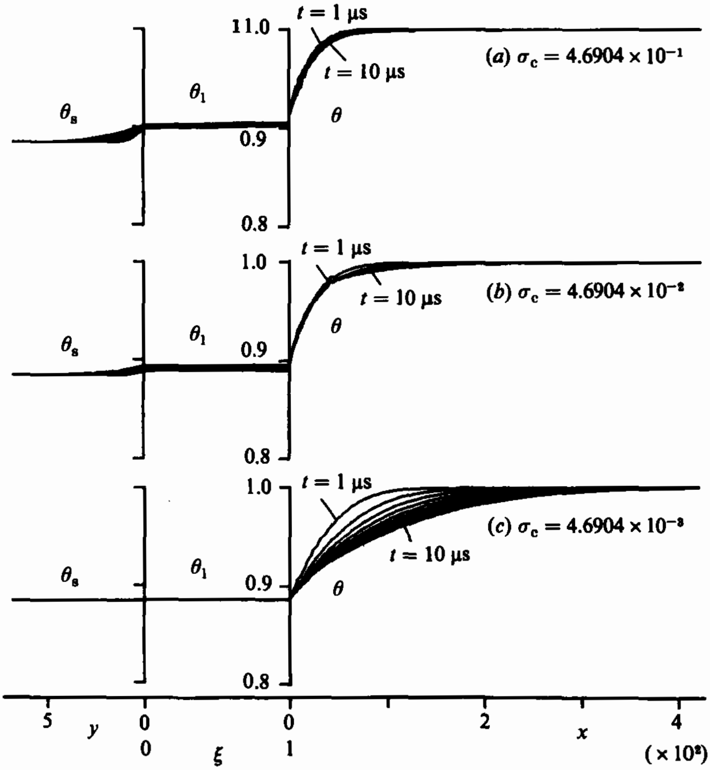

FIGURE 6. Temperature distributions in the vapour, liquid film and shock-tube endwall.

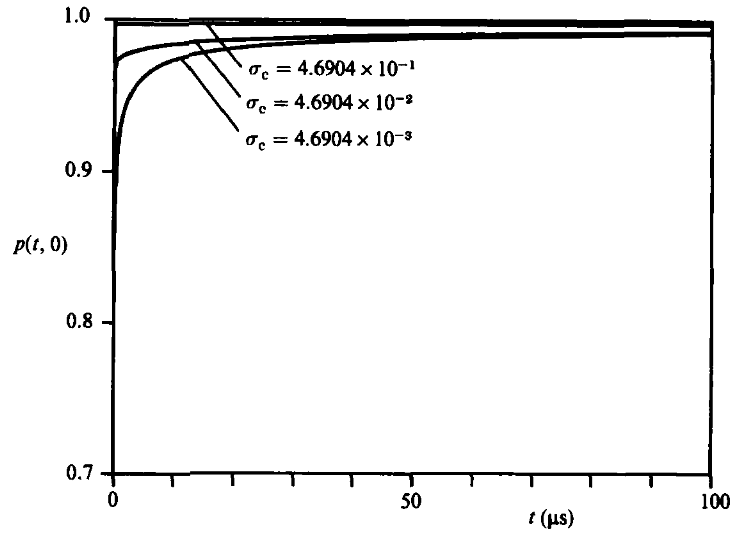

FIGURE 7. The time history of the vapour pressure.

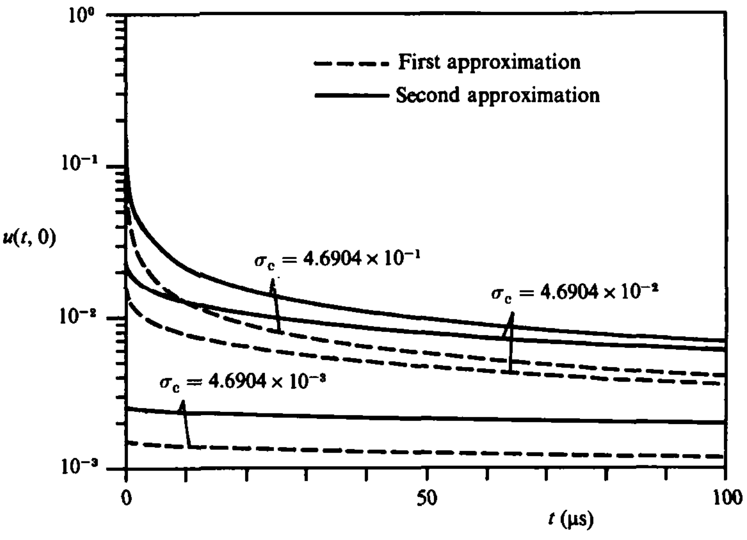

FIGURE 8. The time history of the vapour velocity at the liquid-film surface.

wave because it is very time consuming: only the first perturbation solution of the temperature distribution is computed. The space coordinate in the vapour region is normalized by a mean free path $l_\infty = 16\mu_\infty/(5\rho_\infty(2\pi\mathscr{R}'T_\infty)^{\frac{1}{2}}) = 2.2566 \times 10^{-1}$ μm, in the liquid-film region by an instantaneous thickness of the liquid film, and in the endwall region by a reference quantity $(D'_s\tau)^{\frac{1}{2}} = 7.5750 \times 10^{-1}$ μm ($\tau = 1$ μs). It is found that the larger the value of σ_c, the more remarkable is the development of the thermal boundary layer into the vapour region prevented. This is because the thermal boundary layer develops against the counter flow of the vapour, which becomes larger according to the increase of σ_c. The temperature in the liquid film is found to be spatially uniform. The temperature jump at the vapour–liquid interface is small for all values of σ_c under the present condition.

Figure 7 shows the time history of the vapour pressure (second approximation).

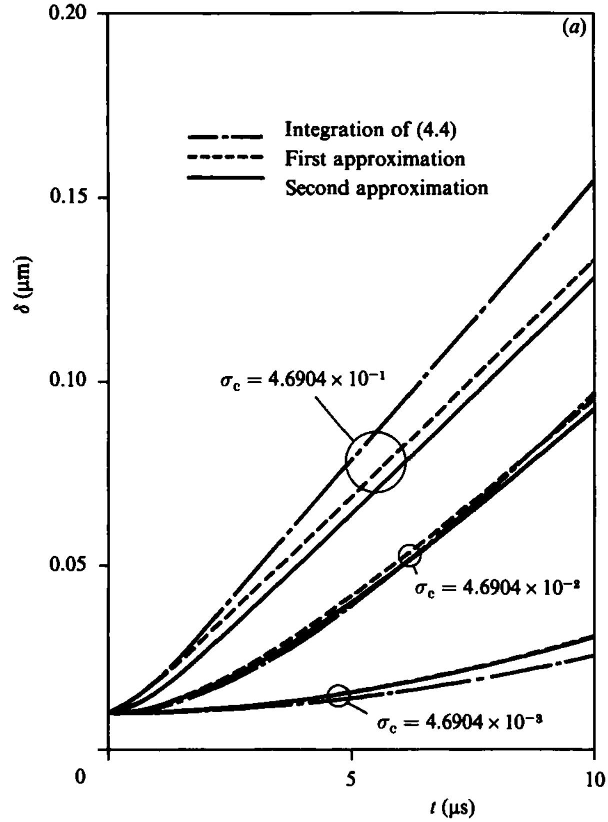

FIGURE 9(*a*). For caption see page 321.

The pressure is normalized by $p_\infty = 1.7786 \times 10^4$ Pa, which is determined from the Rankine–Hugoniot relation. The first approximate solution for the pressure is just unity. At the instant of reflection of the shock wave, the pressure drops rapidly by about 15 % from the Rankine–Hugoniot value for $\sigma_c = 4.6904 \times 10^{-1}$, while it drops only by about 0.3 % for $\sigma_c = 4.6904 \times 10^{-3}$. The pressure drop is mainly caused by the density decrease due to the rapid condensation at the liquid surface (equation (2.140)).

Figure 8 shows the time history of the vapour velocity at the liquid surface. The vapour flow is subsonic for all cases of σ_c and it is greatly influenced by the value of σ_c at the initial time stages.

Figure 9 shows the time history of the thickness of the liquid film on the shock-tube endwall: (*a*) $\sigma_c = 4.6904 \times 10^{-1}$, (*b*) $\sigma_c = 4.6904 \times 10^{-2}$ and (*c*) $\sigma_c = 4.6904 \times 10^{-3}$. The growth behaviour of the liquid film is drastically influenced by the value of σ_c. The growth of the liquid film is characterized as follows:

(i) There exists a transition process from a t-dependent growth of the liquid film to a $t^{\frac{1}{2}}$-dependent one.

(ii) The t-dependent growth is governed by the condensation parameter σ_c, while the $t^{\frac{1}{2}}$-dependent one is independent of it.

(iii) The transition time before the $t^{\frac{1}{2}}$-dependent growth begins becomes shorter with increase of σ_c.

Let us consider in detail the transition mechanism of the liquid-film growth. The

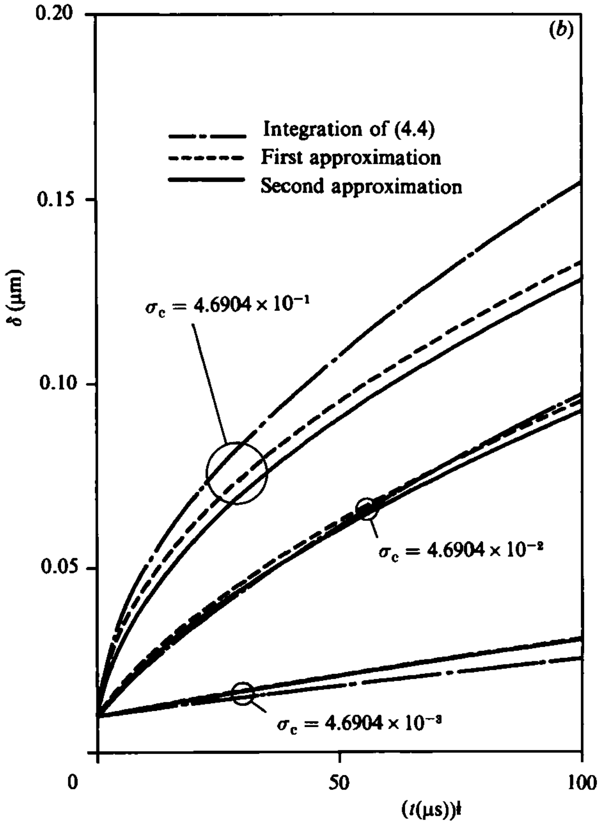

FIGURE 9(b). For caption see facing page.

growth rate of the liquid film can be described by (2.58b), (2.68), (2.91) and (2.97) as follows:

$$\dot{\delta}_0(t) \doteqdot \frac{1}{\rho_l}\frac{p_0^*(0)}{1-p_0^*(0)}\left(\frac{2}{\gamma c}\right)^{\frac12}\{1-d_1\theta_{l0}(t,\xi=1)-d_2\}, \tag{4.1}$$

$$\theta_{l0}(t,\xi=1) \doteqdot \theta_{s0}(t,y=0) \tag{4.2a}$$

$$\doteqdot c+\frac{\rho_l L}{\lambda_s}\left(\frac{D_s}{\pi}\right)^{\frac12}\int_0^t \frac{\dot{\delta}_0(t')}{(t-t')^{\frac12}}\,\mathrm{d}t', \tag{4.2b}$$

where the assumptions of $\rho(t,0) \doteqdot \rho^*(t)$ and $|F| \ll 1$ are made, the saturated vapour pressure is given by $p_0^* = d_1\theta_{l0}+d_2$ ($d_1 = 11.677$ and $d_2 = -9.6520$ for the present condition), and the heat transferred by the heat conduction is neglected because it is much smaller than the latent heat of condensation. Equations (4.1) and (4.2a, b) indicate that the growth rate of the liquid film is influenced by the temperature change of the liquid-film surface and *vice versa*. From (4.1) and (4.2b), we obtain the second kind of Volterra integral equation:

$$\dot{\delta}_0(t) = B-\frac{C}{\pi^{\frac12}}\int_0^t \frac{\dot{\delta}_0(t')}{(t-t')^{\frac12}}\,\mathrm{d}t', \tag{4.3}$$

where

$$B = \frac{(1-d_1c-d_2)}{\rho_l}\left(\frac{2}{\gamma c}\right)^{\frac12}\frac{p_0^*(0)}{1-p_0^*(0)}, \quad C = \frac{d_1 L}{\lambda_s}\left(\frac{2D_s}{\gamma c}\right)^{\frac12}\frac{p_0^*(0)}{1-p_0^*(0)}.$$

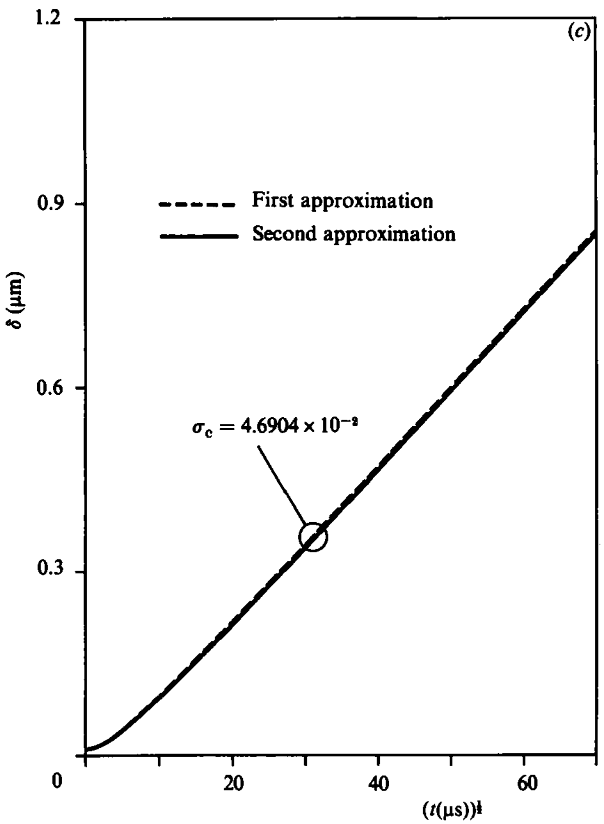

FIGURE 9. The time history of the thickness of the liquid film formed on the shock-tube endwall. (*a*) Variation of the liquid-film thickness versus the time, (*b*) variation of the liquid-film thickness versus the square root of the time and (*c*) variation of the liquid-film thickness versus the square root of the time for a longer period: $\sigma_c = 4.6904 \times 10^{-2}$.

The second term on the right-hand side of (4.3) represents the variation of the growth rate of the liquid film owing to the temperature change of the liquid-film surface. It is found that the growth rate of the liquid film is almost constant at the early stages when the temperature change of the liquid film is small, but it begins to vary as the temperature changes. The solution of (4.3) can be obtained by the Laplace transformation (Erdelyi *et al.* 1954) as

$$\dot{\delta}_0(t) = B \exp(C^2 t) \operatorname{erfc}(C t^{\frac{1}{2}}). \tag{4.4}$$

Numerical solution of the differential equation (4.4) is also shown in figure 9(*a*, *b*). It is in fair agreement with the first and second perturbation solutions, especially in the early time stages.

We will inspect the asymptotic behaviour of (4.4) now. For $Ct^{\frac{1}{2}} \ll 1$, (4.4) is expanded as follows:

$$\dot{\delta}_0(t) \doteqdot B\left\{1 - 2C\left(\frac{t}{\pi}\right)^{\frac{1}{2}} + \dots\right\} \tag{4.5a}$$

$$\doteqdot B. \tag{4.5b}$$

The integrated form of (4.5*b*) is

$$\delta_0(t) \doteqdot Bt+e_1 \tag{4.6a}$$

or, in a dimensional form,

$$\delta_0'(t') \doteqdot a_\infty \sigma_c B\left\{\frac{1}{p_0^*(0)}-1\right\}t'+e_1', \tag{4.6b}$$

where e_1 and e_1' are constants. The latter constant e_1' is an initial thickness of the liquid film. The liquid film grows in linear proportion to the time, and the growth rate of the liquid film is constant if the condensation parameter is constant for a vapour. It should be noted that the growth rate of the liquid film is dependent on the condensation parameter. The surface temperature of the liquid film is then given from (4.2*b*) and (4.5*b*):

$$\theta_{10}(t,\xi=1) = c+\frac{2\rho_1 BL}{\lambda_s}\left(\frac{D_s}{\pi}t\right)^{\frac{1}{2}}. \tag{4.7}$$

The second term on the right-hand side of (4.7) is very small in comparison with the initial endwall temperature c at this stage.

For $Ct^{\frac{1}{2}} \gg 1$, on the other hand, (4.4) is expanded as follows:

$$\dot{\delta}_0(t) \doteqdot \frac{B}{C(\pi t)^{\frac{1}{2}}}\left\{1-\frac{1}{C(\pi t)^{\frac{1}{2}}}+\dots\right\} \tag{4.8a}$$

$$\doteqdot \frac{B}{C(\pi t)^{\frac{1}{2}}}. \tag{4.8 b}$$

The integral form of (4.8*b*) is

$$\delta_0(t) \doteqdot \frac{2B}{C}\left(\frac{t}{\pi}\right)^{\frac{1}{2}}+e_2 \tag{4.9a}$$

or, in a dimensional form,

$$\delta_0'(t') \doteqdot \frac{2B}{C}\left(\frac{D_\infty}{\pi}t'\right)^{\frac{1}{2}}+e_2' \tag{4.9b}$$

where e_2 and e_2' are constants. The latter constant e_2' can be regarded as a liquid-film thickness at the instant when the film begins to grow in proportion to the square root of the time. It should be noted that the growth behaviour of the liquid film is not influenced at all by the condensation parameter. The surface temperature of the liquid film is then given from (4.2*b*) and (4.8*b*) as

$$\theta_{10}(t,\xi=1) \doteqdot \frac{1-d_2}{d_1}-\frac{1-d_1c-d_2}{d_1C(\pi t)^{\frac{1}{2}}} \tag{4.10a}$$

$$\doteqdot \frac{1-d_2}{d_1}. \tag{4.10b}$$

This result indicates that the state in which the condition $Ct^{\frac{1}{2}} \gg 1$ is satisfied can be regarded as a final equilibrium state. The temperature approaches a constant value given by (4.10*b*) at this stage. It is 0.912 for the present condition and it is shown in figure 4(*a*). From (2.95*a*), (4.9*a*) and (4.10*b*), we can obtain the final value of the function F as $t\rightarrow\infty$, i.e. (2.95*b*). This value is 0.025 for the present condition. It is found that the function F has little influence on the temperature of the liquid film. Concerning the effect of the condensation parameter upon the transition time, the larger the condensation parameter, the more rapidly does the surface temperature of the liquid film rise and, in consequence, the more rapidly does the state attain the final equilibrium form.

It should finally be stressed that there exists a transition process of the liquid-film growth from the t-dependent growth to the $t^{\frac{1}{2}}$-dependent one and that the condensation parameter is related to the t-dependent growth. This suggests that the measurement of the condensation parameter by shock tubes should be made while the liquid film is growing in proportion to the time.

5. Conclusions

The condensation phenomenon of a vapour on the endwall of a shock tube behind a reflected shock wave has been investigated by the method of matched asymptotic expansions. The effects of the condensation parameter on the gas dynamics, the thermal process in the liquid film and the shock-tube endwall, and the growth behaviour of the liquid film have been quantitatively predicted. It has been found that there exists a transition process in the growth of the liquid film, that is, the liquid film grows approximately in proportion to the time in the early stages after the reflection of the shock wave and later in proportion to the square root of the time. The transition time from the t-dependent growth to the $t^{\frac{1}{2}}$-dependent one depends principally upon the condensation parameter. In the t-dependent growth region, the growth rate of the liquid film is proportional to the condensation parameter, depending upon an initial condition and the thermal properties of the vapour and the liquid film, while in the $t^{\frac{1}{2}}$-dependent growth one it becomes independent of the condensation parameter and is controlled only by thermal properties of the vapour, liquid film and shock-tube endwall. This result suggests that the measurement of the condensation parameter of vapour by shock tubes should be made in the t-dependent growth region, i.e. in the early stages after the reflection of the shock wave.

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